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PROJECT TITLE: **Mathematics of the Future: Understanding and Application of Mathematics with the help of Technology, FutureMath**

Programme: Erasmus+
Key Action: Cooperation for innovation and the exchange of good practices
Action Type: Strategic Partnerships for higher education
Ref. No.: 2020-1-RS01-KA203-065388

Intellectual Output 5: Analysis and further developmen

Prepared by UNS

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The results:

5.1 **Results of analysis of the impact and effects of pilot courses**
5.2 **Created framework for integrating STEAM principles for STEM studies**

are developed by SIM with the help of UNS, BMU. UPT and GDU

The results of

5.**3 Published online teaching and learning materials for Calculus courses with STREAM revised topics**

are

Topics

1. Partial derivative-in English-UNS
2. Application of partial derivatives-in English-UNS
3. Directional Derivatives -in English-UNS
4. Application of directional Derivatives -in English-UNS
5. Complex functions in Sage math -in English-UNS
6. Combinatorics: Splitting-in English-UNS
7. Combinatorics: Variations, combinations-in English-UNS
8. Elementary functions with Geogebra-in English-UNS
9. Ordinary differential equations-in English-GDU
10. Estimating area-in English- GDU
11. Two Variables-in English- GDU
12. GamePhysics Movement, BMU
13. Classification in machine learning -in English- BMU
14. Definite integrals solving in Java-in English- BMU
15. Elliptic curve cryptography-in English- BMU
16. Metod najmanjih kvadrata –in Serbian-BMU
17. Line integral 1-in Macedonian-SIM
18. Line integral 2-in Macedonian-SIM
19. Double integral -in English-UPT
20. Numerical solution of differential equation -in Romanian-UPT

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Lectures

1. Fizika Igara- in Serbian- BMU
2. Korelaciona i regresivna analiza- in Serbian-BMU
3. Recurence relation -in English-BMU
4. Classification in machine learning -in English- BMU
5. Elliptic curve cryptography-in English- BMU
6. Numerical solution of differential equation -in Romanian-UPT
7. Numerical solution of differential equation -in English-UPT

Video material

1. UNS Solving equations in SageMath, Dragan Mašulović, in Serbian, online, film Complex numbers - Lecture.mp4 - OneDrive (live.com).
2. UNS, Application of derivatives, , online, film UNS-ApplicatDerivat1.mp4 - PMF Nextcloud
3. UPT-- AMIS01, in Romanian, online, film Curs AMIS 01.mp4 - PMF Nextcloud (uns.ac.rs)
4. UPT AMIS02, in Romanian, online, film Curs AMIS 02.mp4 - PMF Nextcloud (uns.ac.rs)
5. UPT AMIS03, in Romanian, on line film, Curs AMIS 03.mp4 - PMF Nextcloud (uns.ac.rs)
6. GDU-STEAM in ordinary differential equation, on line film, IO5-STEAM in Ordinary Dif. Eq. (English).mp4 - PMF Nextcloud (uns.ac.rs)
7. GDU-Primena na izvod, on line film, in Macedonian, Примена на извод на функција со една променлива - втор дел-20220519_170054-Meeting Recording(1) (1).mp4 - PMF Nextcloud (uns.ac.rs)

Tests

Test for checking students' knowledge are presented below.

UNS-Directional derivative

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Test - directional derivatives

1. Let $z = x^2 - 3y$ and let $P = (2,1)$. Find the directional derivative of z, at P, in the following directions:
   - toward the point $Q = (-1,0)$
   - in the direction of $(3,2)$ and
   - toward the origin.
2. Let $z = \sin x + 2\cos y$ and let $P = \left(\frac{\pi}{4}, \frac{\pi}{3}\right)$. Find the directions of maximal/minimal increase, and find a direction where the instantaneous rate of z change is 0.
3. Find the gradient of function $u = x^2 + 2y^2 - 3yz$.

UNS-Application of partial derivative

Applications of partial derivatives - test

1. Find the equation of the tangent plane of the surface $z = 6 - x^2 - y^2$ in the point $A(2, 2, -2)$.
2. In the ideal gas state equation $p = \frac{RT}{V}$ what is the pressure(P) change rate in bars the point $T = 100C$, $V = 20l$ if the volume stays constant.

UNS Test – Primena izvoda po pravcu

1. Odrediti izvod po pravcu $D_{\vec{u}}f$ funkcije $f(x,y) = \cos(x + 3y) + 4x\text{^}2y$ u pravcu jediničnog vektora $\vec{u}$ čiji je argument $\pi/6$.
2. Skicirati vektorsko polje $\vec{F}(x,y) = x^2y\vec{\imath} + 2y)\vec{\jmath}$
3. Odredidi gradijent vectorskog polja za funkciju $f(x,y) = (x^2 + 4y)e^{2x} + \sin(3x - 2y)$.
4. Neka se penjete na visinu vrh planine 1000 m. Nadmorska visina puta pomkojem se penjete određena je funkcijom $f(x,y) = 1000 - 0.02x^2 - 0.4y^2$. Ako se nalazite na koti, na mestu sa koordinatama $(80,40)$, u kom pravcu je najbrža promena kote?

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UNS Test – Površina pomoću integral

1. Data je funkcija $f(\mathrm{x}) = \cos x$.
   a) Nacrtati grafik funkcije;
   b) Odrediti $\int_0^{\pi} f(\mathrm{x})\mathrm{dx}$, $\int_0^{\pi/2} f(\mathrm{x})\mathrm{dx}$;
   c) Odrediti površinu koju funkcija f za klapa sa x −osom na intervalima $[0, \pi]$ i $[0, \pi/2]$.
2. Izračunati površinu i zapreminu kruga poliprečnika $r = 2,$ pomoću integral.
3. Proveriti rezultate pomoću *GeoGebra*-e.

UNS Test – Complex functions

Test

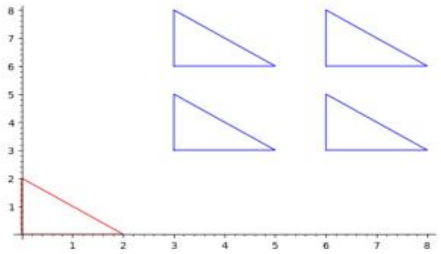
1. (a) Solve the following cubic equation graphically using SageMath:

$$x^3 - 2x^2 + 9x - 18 = 0.$$

   (b) Solve then the equation exactly using SageMath.

2. Using complex arithmetic and SageMath, write Python code that draws a 6-pointed star.

3. Using complex arithmetic and SageMath, write Python code to produce the following



4. Using SageMath visualize the following complex functions:

   (a) $\dfrac{2z-1}{1-z^2}$;

   (b) $|z| + \mathrm{Arg}(z)$;

   (c) $\cos\dfrac{z-1}{z+1}$.

GDU- Application of derivatives – Minimizing and maximizing problems

1. Calculate the first derivative of the next functions:

   a) $f(x) = x^3$ c) $f(x) = \sqrt[3]{x}$ e) $f(x) = \dfrac{3^x}{\sqrt[3]{x}}$

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2. Calculate the first derivative of the next functions:
   a) $f(x)=(x+2)^4$ c) $f(x)=\sqrt{x^2-2x}$

3. Calculate the second derivative of the next functions:
   a) $f(x)=x^2(x-1)$ c) $f(x)=\left(1-\sqrt{x}\right)\sin x$

4. Calculate the local extreme values of the functions:
   a) $f(x)=\frac{x^3}{3}-\frac{x^2}{2}$ c) $f(x)=\frac{x-2}{x+1}$

5. Which two numbers with product 64 give the lowest sum?

GDU-Application of derivatives - Minimizing and maximizing problems

6. Calculate the first derivative of the next functions:
   b) $f(x)=\frac{1}{x^3}$ d) $f(x)=\frac{x^3}{\sqrt[3]{x}}$ f) $f(x)=2^x\cdot\ln x$

7. Calculate the first derivative of the next functions:
   b) $f(x)=\cos(x+2)$ d) $f(x)=\frac{2}{(x-1)^3}$

8. Calculate the second derivative of the next functions:
   b) $f(x)=\left(1+e^x\right)\ln x$ d) $f(x)=\frac{x^3}{3}-\frac{x^2}{2}$

9. Calculate the local extreme values of the functions:
   b) $f(x)=2x^3-3x^2$ d) $f(x)=\frac{x-1}{x^2+2}$

10. Which two numbers with sum 100 give the greatest product?

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GDU-Definition and basic notions for Ordinary Differential Equations (ODE)

Circle the correct answer:

1. The differential equation $y'' = x$ is:
   a) PDE; b) ODE.

2. The differential equation $y'' = x$ is of order:
   a) 3; b) 1; c) 2.

3. The general solution of the differential equation $y'' = x$ contain:
   a) one integration constant;
   b) two integration constants;
   c) three integration constants.

4. The solution $y = \frac{x^2}{2} + C$ of the differential equation $y' = x$ is:
   a) a general solution; b) a particular solution.

5. The solution $y = \frac{x^2}{2}$ of the differential equation $y' = x$ is:
   a) a general solution; b) a particular solution.

Answer the questions:

1. Definition for ODE.
2. What is the difference between the general and the particular solution of the differential equation? Explain geometric visualization of the general and the particular solution for a differential equation!

3. What are the steps for the solving of the task from applied mathematics, natural sciences, and technology?

GDU-Ordinary Differential Equations of first order

Circle the correct answer:

1. Which type is the differential equation $y' = f(\frac{y}{x})$?
   a) Differential equation with separable variables;

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b) Homogeneous differential equation

2. The general solution of the differential equation $y' = x$ contains:
a) one integration constant;
b) two integration constants;
c) three integration constants.

3. Which type is the differential equation $y' = x$?
a) Differential equation with separable variables;
b) Homogeneous differential equation

4. The solution $y = \frac{x^2}{2}$ of the differential equation $y' = x$ is:

a) general solution; b) particular solution.

Answer the questions:

4. Definition of differential equation of first order.

5. What is the difference between the general and the particular solution of the differential equation? Explain geometric visualization of the general and the particular solution for a differential equation!

GDU-functions with two variables – application of partial derivatives

1. Calculate partial derivatives of first and second order for the next functions:
   c) $z = x \ln 2y$ b) $z = x^2\sqrt{y}$ c) $z = \sqrt[3]{xy}$
2. Calculate partial derivatives of first and second order for the next functions:
   a) $z = \ln\frac{2x}{3y}$ b) $z = x + y + 4 + 4\sin x \sin y$
3. Determine the extreme values of the next functions:
   a) $f(x,y) = \frac{x^2}{2} + \frac{y^3}{3} + 3xy$ b) $z = x^2 - xy + y^2 - 2x - 2y$

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4. Determine the dimensions of cuboid with volume 521 liters, such that it will have as small as possible area.

UPT Line integral

1. Compute the value of the integral $\int_C x^2 yz\,ds$, where C is the line segment joining the points (-1,4,0) and (1,5,3).

2. Compute the value of the integral $\int_C x^2+y^2+z^2 ds$, where C has the parametric representation $x=\cos(3t), y=t, z=\sin(3t), t\in[0,2\pi]$.

3. Compute the value of the integral $\int_C x^3 dx+y^3 dy$, where C is the contour of the ABC triangle, A(-1,0), B(0,1), C(1,0).

UPT Double Integral

Compute the integral $\iint_D \frac{dx}{x^2-3x-4}$, where $D = \left[\frac{\pi}{4}, \frac{\pi}{3}\right] \times [1,2]$.

Compute the integral $\iint_D \frac{2ydxdy}{x^2-3x-4}$, where D is the domain between the parabola $y = x^2 - 4x$ and the line $y = x - 4$.

UPT Numerical solution of ODE

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1. We consider the Cauchy problem $\begin{cases} y' = 2 \cdot x \\ y(0) = 1 \end{cases}$. Find the exact (analytical) solution, compute a numerical solution using Euler's method and compare the two solutions.

2. We consider the Cauchy problem $\begin{cases} x' = \frac{2}{t} \cdot x + t^2 \cdot cos(t) \\ x(\pi) = 0 \end{cases}$. Compute and plot on the $[\pi, 2 \cdot \pi]$ interval the exact solution, the numerical solution given by Euler's method and the numerical solution given by the "ode45" method.

3. Compute and plot on the $[0,\ 2 \cdot \pi]$ interval a numerical solution of the Cauchy problem: $\begin{cases} x_1' = -x_2 \\ x_2' = x_1 \\ x_1(0) = 1 \\ x_2(0) = 0 \end{cases}$.
What are the analytical expressions of the functions x_1 and x_2?

UPT Numerical solution of ODE in Romanian

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1. Considerăm problema Cauchy $\begin{cases} y' = 2 \cdot x \\ y(0) = 1 \end{cases}$. Găsiţi soluţia exactă (analitică), calculaţi o soluţie aproximativă (numerică) folosind metoda lui Euler şi comparaţi cele două soluţii.

2. Considerăm problema Cauchy $\begin{cases} x' = \frac{2}{t} \cdot x + t^2 \cdot cos(t) \\ x(\pi) = 0 \end{cases}$. Calculaţi şi reprezentaţi grafic pe intervalul $[\pi, 2 \cdot \pi]$ soluţia exactă (analitică), o soluţie aproximativă (numerică) dată de metoda lui Euler şi o soluţie aproximativă (numerică) dată de comanda "ode45" din Matlab.

3. Calculaţi şi reprezentaţi grafic pe intervalul $[0, 2 \cdot \pi]$ o soluţie aproximativă (numerică) a problemei Cauchy : $\begin{cases} x_1' = -x_2 \\ x_2' = x_1 \\ x_1(0) = 1 \\ x_2(0) = 0 \end{cases}$.
Care sunt expresiile analitice ale funcţiilor x_1 şi x_2?

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