TOPIC PLAN		
Partner organization	UNS	
Topic	Function of Several Variables	
Lesson title	Directional Derivatives	
Learning objectives	✓ Students will be able to determine directional derivatives of functions of several variables, gradient; ✓ Students will acquire and deal with derivatives of a function; ✓ Students will be able to deal with different problems in everyday life, which require finding directional derivatives of a given function; ✓ Students are encouraged to use technology and different software in their work, while considering problem-based situations.	**Strategies/Activities** ☐Graphic Organizer ☐ Think/Pair/Share ☐Modeling ☐Collaborative learning ☐Discussion questions ☐Project based learning ☐Problem based learning
Aim of the lecture / Description of the practical problem	The aim of the lecture is to make students able to understanda the directional derivatives. The teacher gives the next problem to the students: 1. Find each of the directional derivatives a) $\mathrm{D}_{\vec{\mathrm{u}}}\mathrm{f}(2,0)$ where $f(x,y) = xe^{xy} + y$ and $\vec{\mathrm{u}}$ is the unit vector in the direction of $\theta = 2\ \frac{\pi}{3}$. b) $\mathrm{D}_{\vec{\mathrm{u}}}\mathrm{f}(x,y,z)$ where $f(x,y,z) = x^2z + y^3z^2 - xyz$ in the direction of $\vec{\mathrm{v}} = (-1,0,3)$. 2. Suppose that the height of a hill above sea level is given by $z = 1000 - 0.01x^2 - 0.02y^2$. If you are at the point $(60,100)$ in what	**Assessment for learning** ☐Observations ☐Conversations ☐Work sample ☐Conference ☐Check list ☐Diagnostics

<table>
<tr>
<td></td>
<td>direction is the elevation changing fastest? What is the maximum rate of change of the elevation at this point?</td>
<td rowspan="3">Assessment as learning
☐Self-assessment
☐Peer-assessment
☐Presentation
☐Graphic Organizer
☐Homework

Assessment of learning
☐Test
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☐Project
☐Published work</td>
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<tr>
<td>Previous knowledge assumed:</td>
<td>- functions
- algebraic equations
- differentiating techniques</td>
</tr>
<tr>
<td>Introduction / Theoretical basics</td>
<td>Partial derivatives give us an understanding of how a surface changes when we move in the x and y directions. We made the comparison to standing in a rolling meadow and heading due east: the amount of rise/fall in doing so is comparable to f_x. Likewise, the rise/fall in moving due north is comparable to f_y. The steeper the slope, the greater in magnitude f_y. But what if we didn't move due north or east? What if we needed to move northeast and wanted to measure the amount of rise/fall? Partial derivatives alone cannot measure this. This section investigates directional derivatives, which do measure this rate of change. We begin with a definition.

DEFINITION
Let $z = f(x, y)$ be continuous on an open set S and let $\vec{u} = (u_1, u_2)$ be a unit vector. For all points (x, y) the directional derivative of f at (x, y) in the direction of $\vec{u}$ is
$$D_{\vec{u}}f(x,y) = \lim_{h \to 0} \frac{f(x + hu_1, y + hu_2) - f(x,y)}{h}$$
The partial derivatives f_x and f_y are defined with similar limits, but only x or y varies with h, not both. Here both x and y vary with a weighted h, determined by a particular unit vector $\vec{u}$. This may look a bit intimidating but in reality it is not too difficult to deal with; it often just requires extra algebra. However, the following theorem reduces this algebraic load.</td>
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THEOREM Let $z = f(x,y)$, and let $\vec{u} = (u_1, u_2)$ be a unit vector. The directional derivative of f at (x_0, y_0) in the direction of $\vec{u}$ is

$$D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0) u_1 + f_y(x_0, y_0) u_2$$

EXAMPLE 1 Let $z = 14 - x^2 - y^2$ and let $P = (1,2)$. Find the directional derivative of f, at P, in the following directions:

1. toward the point $Q = (3,4)$
2. in the direction of (2,-1) and
3. toward the origin.

Solution We find that $f_x(x,y) = -2x$ and $f_x(1,2) = -2$; $f_y(x,y) = -2y$ and $f_y(1,2) = -4$.

1. Let $\vec{u}$ be the unit vector that points from the point (1,2) to the point $Q = (3,4)$. The vector $\overrightarrow{PQ} = (2,2)$; the unit vector in this direction is $\vec{u} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Thus, the directional derivative of f at (1,2) in the direction of $\vec{u}$ is

$$D_{\vec{u}} f(1,2) = -2\frac{1}{\sqrt{2}} - 4\frac{1}{\sqrt{2}} = -\frac{6}{\sqrt{2}} \approx -4.24$$

Thus the instantaneous rate of change in moving from the point (1,2,9) on the surface in the direction of $\vec{u}$ (which points toward the point Q) is about −4.24. Moving in this direction moves one steeply downward.

2. We seek the directional derivative in the direction of (2,-1). The unit vector in this direction is $\vec{u} = (\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}})$. Thus the directional derivative of f at (1,2) in the direction of $\vec{u}$ is

$$D_{\vec{u}} f(1,2) = -2\frac{2}{\sqrt{5}} - 4\frac{-1}{\sqrt{5}} = 0$$

Starting on the surface of f at (1,2) and moving in the direction of (2,-1) results in no instantaneous change in z-value. This is analogous to standing on the side of a hill and choosing a direction to walk that does not change the elevation. One neither walks up nor down, rather just "along the side'' of the hill.

Finding these directions of "no elevation change'' is important.

3. At $P=(1,2)$, the direction towards the origin is given by the vector $(-1,-2)$; the unit vector in this direction is $\vec{u}=(-\frac{1}{\sqrt{5}},-\frac{2}{\sqrt{5}})$. The directional derivative of fat P in the direction of the origin is

$$D_{\vec{u}}f(1,2)=-2\frac{-1}{\sqrt{5}}-4\frac{-1}{\sqrt{5}}=\frac{10}{\sqrt{5}}\approx 4.47$$

Moving towards the origin means "walking uphill'' quite steeply, with an initial slope of about 4.47.

As we study directional derivatives, it will help to make an important connection between the unit vector $\vec{u}=(u_1,u_2)$ that describes the direction and the partial derivatives f_x and f_y. We start with a definition and follow this with a Key Idea.

DEFINITION Let $z=f(x,y)$ be differentiable on an open set S that contains the point (x_0,y_0).
The **gradient of** f is $\nabla f(x,y)=(f_x(x,y),f_y(x,y))$.
The **gradient of** f at (x_0,y_0) is $\nabla f(x_0,y_0)=(f_x(x_0,y_0),f_y(x_0,y_0))$.

Note: The symbol "∇" is named "nabla," derived from the Greek name of a Jewish harp. Oddly enough, in mathematics the expression ∇f is pronounced "del f".

To simplify notation, we often express the gradient as $\nabla f=(f_x,f_y)$.The gradient allows us to compute directional derivatives in terms of a dot product.

The gradient and Directional Derivatives

The directional derivatives of $z=f(x,y)$ in the direction $\vec{u}$ is

$$D_{\vec{u}}f = \nabla f \cdot \vec{u}$$

The properties of the dot product previously studied allow us to investigate the properties of the directional derivative. Given that the directional derivative gives the instantaneous rate of change of z when moving in the direction of $\vec{u}$ three questions naturally arise:

1. In what direction(s) is the change in z the greatest (i.e., the "steepest uphill'')?
2. In what direction(s) is the change in z the least (i.e., the "steepest downhill'')?
3. In what direction(s) is there no change in z?

Using the key property of the dot product, we have

$$\nabla f \cdot \vec{u} = ||\nabla f|| \cdot ||\vec{u}|| \cdot \cos\theta = ||\nabla f|| \cdot \cos\theta \qquad (*)$$

where θ is the angle between the gradient and $\vec{u}$. (Since $\vec{u}$ is a unit vector, $||\vec{u}|| = 1$.) This equation allows us to answer the three questions stated previously.

1. Equation (*) is maximized when $\cos\theta = 1$, i.e., when the gradient and $\vec{u}$ have the same direction. We conclude the gradient points in the direction of greatest z change.
2. Equation (*) is minimized when $\cos\theta = -1$, i.e., when the gradient and $\vec{u}$have opposite directions. We conclude the gradient points in the opposite direction of the least z change.
3. Equation (*) is 0 when $\cos\theta = 0$, i.e., when the gradient and $\vec{u}$are orthogonal to each other. We conclude the gradient is orthogonal to directions of no z change.

This result is rather amazing. Once again imagine standing in a rolling meadow and face the direction that leads you steepest uphill. Then the direction that leads steepest downhill is directly behind you, and side-stepping either left or right (i.e., moving perpendicularly to the direction you face) does not change your elevation at all.

Recall that a level curve is defined by a path in the $xy-$plane along which the z-values of a function do not change; the directional derivative in the direction of a level curve is 0. This is analogous to walking along a path in the rolling meadow along which the elevation does not change. The gradient at a point is orthogonal to the direction where the z does not change; i.e., the gradient is orthogonal to level curves.

Recall that a level curve is defined as a curve in the xy-plane along which the z-values of a function do not change. Let a surface $z = f(x, y)$ be given, and let's represent one such level curve as a vector--valued function, $\vec{r}(t) = (x(t), y(t))$. As the output of f does not change along this curve, $f\big(x(t), y(t)\big) = c$ for all t, for some constant c.

Since f is constant for all t, $\frac{df}{dt} = 0$. By the Multivariable Chain Rule, we also know

$$\frac{df}{dt} = f_x(x, y)x'(t) + f_y(x, y)y'(t)$$

$$=(f_x(x, y), f_y(x, y)) \cdot (x'(t), y'(t))$$

$$=\nabla f \cdot \vec{r}'(t) = 0$$

This last equality states $\nabla f \cdot \vec{r}'(t) = 0$: the gradient is orthogonal to the derivative of $\vec{r}$, meaning the gradient is orthogonal to $\vec{r}$ itself. Our conclusion: at any point on a surface, the gradient at that point is orthogonal to the level curve that passes through that point.

We restate these ideas in a theorem, then use them in an example.

THEOREM Let $z = f(x, y)$ be differentiable on an open set S with gradient ∇f, let $P = (x_0, y_0)$ be a point in S and let $\vec{u}$ be a unit vector.

1.The maximum value of $D_{\vec{u}}f\big(x_0, y_0\big)$ is $||\nabla f\big(x_0, y_0\big)||$; the direction of maximal z increase is $\nabla f\big(x_0, y_0\big)$

2. The minimum value of $D_{\vec{u}}f\big(x_0, y_0\big)$ is $-||\nabla f\big(x_0, y_0\big)||$; the direction of minimal z increase is $-\nabla f\big(x_0, y_0\big)$

3. At P, $\nabla f(x_0, y_0)$ is orthogonal to the level curve passing through $(x_0, y_0, f(x_0, y_0))$

EXAMPLE *Finding directions of maximal and minimal increase*

Let $f(x,y) = \sin x \cos y$ and let $P = \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$. Find the directions of maximal/minimal increase, and find a direction where the instantaneous rate of z change is 0.

Solution We begin by finding the gradient. $f_x = \cos x \cos y$ and $f_y = -\sin x \sin y$, thus $\nabla f = (\cos x \cos y, -\sin x \sin y)$ and, at P, $\nabla f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = (\frac{1}{4}, -\frac{3}{4})$

Thus the direction of maximal increase is $(\frac{1}{4}, -\frac{3}{4})$. In this direction, the instantaneous rate of z change is 0.79.

The direction of minimal increase is $(-\frac{1}{4}, \frac{3}{4})$; in this direction the instantaneous rate of z change is -0.79

Any direction orthogonal to ∇f is a direction of no z change. We have two choices: the direction of $(3{,}1)$ and the direction of (-3,-1). The unit vector in the direction of $(3{,}1)$ is shown in each graph of the figure as well. The level curve at $z = \sqrt{3}/4$ is drawn: recall that along this curve the z -values do not change. Since $(3{,}1)$ is a direction of no z -change, this vector is tangent to the level curve at P.

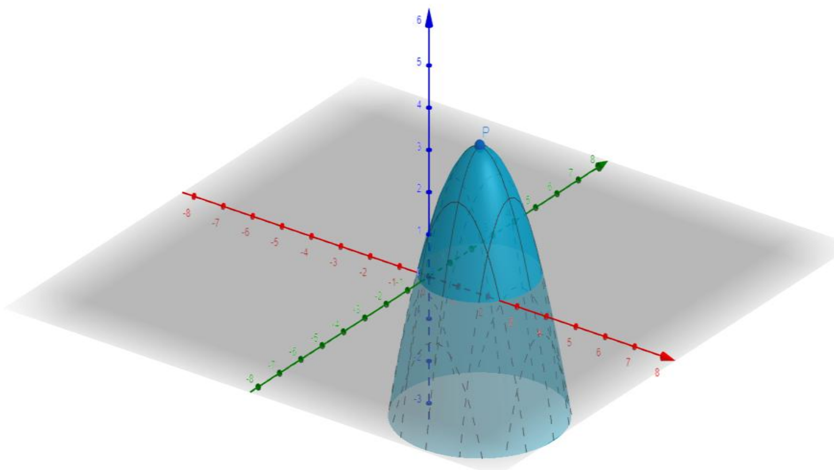
EXAMPLE Understanding when $\nabla f = \vec{0}$.

Let $f(x,y) = -x^2 + 2x - y^2 + 2y + 1$. Find the directional derivative of f in any direction at $P(1{,}1)$.

Solution We find $\nabla f = (-2x + 2, -2y + 2)$. At P, we have $\nabla f(1{,}1) = (0{,}0)$. According to Theorem before, this is the direction of maximal increase. However, $(0{,}0)$ is directionless; it has no displacement. And regardless of the unit vector $\vec{u}$ chosen, $D_{\vec{u}} f - 0$.

We can see that P lies at the top of a paraboloid. In all directions, the instantaneous rate of change is 0.

So what is the direction of maximal increase? It is fine to give an answer of $\vec{0} = (0,0)$, as this indicates that all directional derivatives are 0.



The fact that the gradient of a surface always points in the direction of steepest increase/decrease is very useful, as illustrated in the following example.

EXAMPLE The flow of water downhill

Consider the surface given by $f(x, y) = 20 - x^2 - 2y^2$. Water is poured on the surface at (1,1/4). What path does it take as it flows downhill?

Solution Let $\vec{r}(t) = (x(t), y(t))$ be the vector-valued function describing the path of the water in the xy −plane; we seek $x(t)$ and $y(t)$. We know that water will always flow downhill in the steepest direction; therefore, at any point on its path, it will be moving in the direction of $-\nabla f$. We ignore the physical effects of momentum on the water. Thus $\vec{r}(t)$ will be parallel to

∇f, and there is some constant c such that $c\ \nabla \mathrm{f} = \vec{\mathrm{r}}'(\mathrm{t}) = \left(\mathrm{x}'(\mathrm{t}), \mathrm{y}'(\mathrm{t})\right)$.
We find $\nabla \mathrm{f} = (-2\mathrm{x}, -4\mathrm{y})$ and write $\mathrm{x}'(\mathrm{t})$ as $\frac{dx}{dt}$ and $\mathrm{y}'(\mathrm{t})$ as $\frac{dy}{dt}$. Then

$$(-2cx, -4cy) = (\frac{dx}{dt}, \frac{dy}{dt})$$

This implies $c = -\frac{1}{2x}\frac{dx}{dt}$ and $c = -\frac{1}{4y}\frac{dy}{dt}$. As c equals both expressions, we have $\frac{1}{2x}\frac{dx}{dt} = \frac{1}{4y}\frac{dy}{dt}$. To find an explicit relationship between x and y, we can integrate both sides with respect to t. Recall from our study of differentials that $\frac{dx}{dt}dt = dx$. Thu

$$\int \frac{1}{2x}\frac{dx}{dt}dt = \int \frac{1}{4y}\frac{dy}{dt}dt$$
$$\int \frac{1}{2x}dx = \int \frac{1}{4y}dy$$
$$\frac{1}{2}\ln|x| = \frac{1}{4}\ln|y| + C_1$$

$$2\ln|x| = \ln|y| + C_1$$
$$\ln|x^2| = \ln|y| + C_1$$

Now raise both sides as a power of e
$$x^2 = e^{\ln|y| + C_1}$$
$$x^2 = e^{\ln|y|}e^{C_1}$$
$$x^2 = yC_2$$
$$y = Cx^2$$
As the water started at the point (1,1/4), we can solve for C:
$$C = \frac{1}{4}.$$
Thus the water follows the curve $y = x^2/4$ in the xy-plane.

Functions of Three Variables

The concepts of directional derivatives and the gradient are easily extended to three (and more) variables.

DEFINITION Let $w = F(x, y, z)$ be differentiable on an open ball B

and let $\vec{u}$ be a unit vector in R^3.

The gradient of F is $\nabla F = (F_x, F_y, F_z)$.

The directional derivative of F in the direction of $\vec{u}$ is $D_{\vec{u}}F = \nabla F \cdot \vec{u}$.

The same properties of the gradient when f is a function of two variables, hold for F, a function of three variables.

THEOREM Let $w = F(x, y, z)$ be differentiable on an open ball B, with gradient ∇F, and let $\vec{u}$ be a unit vector.

1.The maximum value of $D_{\vec{u}}F$ is $||\nabla F||$, obtained when the angle between ∇F and $\vec{u}$ is 0, i.e., the direction of maximal increase is ∇F

2. The minimum value of $D_{\vec{u}}F$ is $-||\nabla F||$, obtained when the angle between ∇F and $\vec{u}$ is π, i.e., the direction of minimal increase is $-\nabla F$

3. $D_{\vec{u}}F=0$ when ∇F and $\vec{u}$ are orthogonal.

We interpret the third statement of the theorem as "the gradient is orthogonal to level surfaces,'' the three-variable analogue to level curves.

EXAMPLE *Finding directional derivatives with functions of three variables*

If a point source S is radiating energy, the intensity I at a given point P in space is inversely proportional to the square of the distance between S and P. That is, when $S = (0,0,0), I(x, y, z) = \frac{k}{x^2+y^2+z^2}$, for some constant k.

Let $k = 1$, let $\vec{u} = (\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$ be a unit vector, and let $P = (2,5,3)$. Measure distances in inches. Find the directional derivative of I at P in the direction of $\vec{u}$, and find the direction of greatest intensity increase at P.

Solution We need the gradient ∇I, meaning we need I_x, I_y, I_z. Each partial derivative requires a simple application of the Quotient Rule, giving

$$\nabla I = \left(\frac{-2x}{(x^2+y^2+z^2)^2}, \frac{-2y}{(x^2+y^2+z^2)^2}, \frac{-2z}{(x^2+y^2+z^2)^2}\right)$$

	$\nabla \mathrm{I}(2,5,3) = (\frac{-4}{1444}, \frac{-10}{1444}, \frac{-6}{1444})$ $\mathrm{D}_{\vec{\mathrm{u}}}\mathrm{I} = \nabla \mathrm{I}(2,5,3) \cdot \vec{\mathrm{u}} = \frac{-17}{2166}$ The directional derivative tells us that moving in the direction of $\vec{\mathrm{u}}$ from *P* results in a decrease in intensity of about −0.008 units per inch. (The tensity is decreasing as $\vec{\mathrm{u}}$ moves one farther from the origin than *P*.) The gradient gives the direction of greatest intensity increase. Notice that $\nabla \mathrm{I}(2,5,3) = \frac{2}{1444}(-2,-5,-3)$ That is, the gradient at (2,5,3) is pointing in the direction of (−2,−5,−3), that is, towards the origin. That should make intuitive sense: the greatest increase in intensity is found by moving towards to source of the energy. The directional derivative allows us to find the instantaneous rate of *z* change in any direction at a point. We can use these instantaneous rates of change to define lines and planes that are *tangent* to a surface at a point, which is the topic of the next section.	
Action		

<table>
<tr><td>Materials / equipment / digital tools / software</td><td><u>The materials for learning</u> are given as a part of references of the end from this topic plan;
<u>Equipment</u>: classroom, whiteboard, marker in different colours;
<u>Digital tools</u>: laptop, projector;
<u>Software</u>: Geogebra</td><td></td></tr>
<tr><td>Consolidation</td><td colspan="2">With the given examples students can consider that the real functions and their derivatives are important for solving real life problems. Students will learn what is a directional derivative of a function and gradient and how to calculate it. They can learn how to apply directional derivatives in real problem. Students can use technology, different digital tools and software as a help for solving problems, but can also realize that even with technology, solving different everyday problems is difficult without math knowledge.</td></tr>
<tr><td colspan="3">Reflections and next steps</td></tr>
<tr><td colspan="2">Activities that worked</td><td>Parts to be revisited</td></tr>
<tr><td colspan="2">Problem solving, collaboration, using technology</td><td>Depends on the students, in a conversation with students the teacher will realize the difficulties that students had and then revisit appropriate parts.</td></tr>
<tr><td colspan="3">References</td></tr>
<tr><td colspan="3">[1] J. Stewart, Calculus, Thomson Learning, China, 2006.
[2] M. L. Bittinger, D. J. Ellenbogen and S.A. Surgent, “Calculus and its applications”, Addison-Wesley, 2012.
[3] T. Došenović, A. Takači, D. Rakić, Udžbenik iz Matematike II za studente Tehnološkog fakulteta, Univerzitet u Novom Sadu, 2017.</td></tr>
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