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Intellectual Output 5: Analysis and further development

Result 5.1 Results of analysis of the impact and effects of pilot courses;

Result 5.2 Created framework for integrating STEAM principles for STEM

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Prepared by SIM

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1. INTRODUCTION

Since Newton, we have become used to science making confident predictions about the future, including on the motion of the planets and the times of the tides. However, some things seem very hard to predict, such as the stock market or the weather. Is this a fault in the way we model these systems, or is there a genuine limit to how far we can predict the future? One explanation comes from the theory of chaos, which explains why small changes now can lead to large uncertainty in the future.

Mathematics education is changing rapidly and a big driver for this is the use of new technology. In particular the widespread use of computers has transformed the way that we do mathematics, with computers not only able to mark exam papers, but also to do the algebra required to answer the questions.

Most of mathematics education is supported by empirical papers. In the 1970s, most research was quantitative, and data was used to “prove” that a given method of teaching was better than another. Empirical data had the same role it plays to this day in a good part of what is considered science: there were control groups and experimental groups, and the methodology was based on (or reduced to) statistical treatment and conclusions. Later last century, and earlier this century, qualitative research has swung the pendulum in another direction. Qualitative research sees data as a voice, as a complement that should be added to other evidence in order to make (“prove”) a point.

An analysis such as this one serves the purpose of discussing ideas and presenting bases for research papers, so that we can know the new trends in mathematics education, or as we prefer within this project the future of mathematical education, in the different epistemological positions that characterize our community.

1.1 Recent Perspectives on Mathematical Undergraduate Teaching and Learning.

This analysis addresses a crucial challenge to changing and improving mathematical undergraduate education based on the results we gain trough the Erasmus plus project we have been involved in. The challenge we took was big one: how to increase the effectiveness of undergraduate teaching in mathematics in ways that will follow modern trends of technology, and new ways of teaching methodologies. How to enable faculty to enhance student learning, and continually improve teaching in mathematics. We consider these issues to be the most significant challenge since many undergraduate faculty have received little or no formal training in techniques or strategies for teaching effectively, assessing student learning, or evaluating the effectiveness of their own teaching or that of their colleagues.

Such training is not a firm requirement for being hired as a college-level faculty member. Formal, ongoing programs for professional development aimed at improving teaching are still rare at many postsecondary institutions.

Maybe the most important thing we will deal with in this analysis is how the students respond to the new way of teaching, do they support it and find it useful, and how the knowledge they gained can be used in their further education and future jobs.

1.2 Characterizing an Effective mathematical Undergraduate Teaching.

Graduate students, faculty, and administrators from all types of postsecondary institutions in Europe are increasingly interested in the revamping of teaching practices to enhance student learning in mathematics. Some faculty and departments are confronting the pedagogical and infrastructural challenges of offering smaller classes (e.g., the need for additional teachers to teach more sections), especially for Calculus courses. Others are using innovative approaches to teaching based on emerging research in the cognitive and brain sciences about how people learn. Still others are experimenting with the effectiveness of different learning strategies to accommodate the broader spectrum of students that will enroll mathematics or STEM studies.

Within this project we tried to find a generalized framework for integrating STEM principles into STEM subjects. We have developed, and test it on the pilot lessons a new teaching methodology. One of the first thing we have considered when developing the new teaching methodology is (as we call it) effective teaching. We believe that following items are relevant for the effective teaching:

- Knowledge of subject matter.

Teachers must remain active in their areas of scholarship to ensure that the content of their courses are current, accurate, and balanced, especially when presenting information that may be open to alternative interpretation or disagreement by experts in the field. They also should appreciate the connection between sciences and different fields of research.

Only teachers with great knowledge of the subject matter can help students learn and understand the general principles of their discipline. They are able to provide students with an overview of the whole domain of the discipline, they possess sufficient knowledge and understanding of their own and related sub-disciplines to answer most students' questions and know how to help students find appropriate information, and they are genuinely interested in what they are teaching.

- Skill, experience, and creativity with a range of appropriate pedagogies and technologies.

Deep understanding of subject matter is critical to excellent teaching, but not sufficient. Effective teachers also understand that, over the course of their educational experiences, undergraduates develop different strategies for maximizing their individual abilities to learn, reason, and think critically about complex issues. To be most effective, teachers need to employ a variety of learning strategies and contextually appropriate pedagogies that serve the range of students' learning styles.

They are organized and communicate clearly to students their expectations for learning and academic achievement. They focus on whether students are learning what is being taught and view the learning process as a joint venture between themselves and their students. They encourage discussion and promote active learning strategies. They have the ability to recognize students who are not achieving to their fullest potential and then employ the professional knowledge and skill necessary to assist them in overcoming academic difficulties.

As information and other technologies become more pervasive in teaching and learning of the natural sciences, mathematics, and engineering, a faculty member's use of such resources is likely to become an increasingly important component of teaching evaluations. As with other areas of pedagogy in which college-level faculty have had little formal training or professional development, they will have to learn appropriate and effective uses of hardware and software that are coupled with new ways of viewing teaching and learning.

- Understanding of and skill in using appropriate assessment practices.

Learning of the students should be assess in ways that are consistent with the objectives of a course and integrate stated course objectives with long-range curricular goals. Teachers should determine accurately and fairly students' knowledge of the subject matter and the extent to which learning has occurred throughout the term (not just at the end of the course).

- Professional interactions with students within and beyond the classroom.

Faculty meet with all classes and assigned teaching laboratories, post and keep regular office hours, and hold exams as scheduled. They demonstrate respect for students as individuals; this includes respecting the confidentiality of information gleaned from advising or student conferences. They encourage the free pursuit of learning and protect students' academic freedom. They address sensitive subjects or issues in ways that help students deal with them maturely. They contribute to the ongoing intellectual development of individual students and foster confidence in the students' ability to learn and discover on their own. They uphold and model for students the best scholarly and ethical standards.

2. MATHEMATICS TECHNOLOGY

Lately, many studies on the effects of the technology supported learning are done, see [1, 5, 7]. Within these studies both potential advantages and risks from the use of technology are considered and analyzed. In 2016 OECD made a report on innovation in education concerning the power of digital technologies, see [6], where they clearly stated that new technologies can facilitate innovative pedagogical models, simulation and virtual online laboratories, international collaboration, real time formative assessment and e-learning.

New technology-enhanced educational models present not so much a technological challenge or cost challenge but a pedagogic challenge. To adopt these new models requires teachers to revisit their pedagogy and this may amount to the greatest cost and challenge. The efficacy of the technology-supported models does not come from technology alone, but from the pedagogy that it supports. Without good pedagogic resources and a good understanding of how to use technology to foster deeper learning, these models may not yield the expected outcomes.

Advances in the capabilities and user-friendliness of mathematical software mean that a whole range of problems which previously would have needed graduate level skills to solve can now be accessed by first year undergraduates. For example, to solve an integral in Mathematica these days is enough to know the Integrate command, and the syntax in this software. There is no need for complex rules of the integration process. The question that arises is: What we consider to be a technology for mathematical education, or "mathematics technology?"? In this set we can put the following items:

- Pocket calculators with different symbolic and/or numerical and/or graphical capabilities.
- Mathematical computer programs: symbolic and numerical ones, e.g. Computer Algebra Systems (CAS) like Maple, Mathematica, or MathCad.
- Numerical programs like Matlab.
- Dynamic geometry programs like Cabri and Geogebra.
- Spreadsheet programs.
- Engineering programs based on mathematical models.

The effect of computer technology on education seems to be greater in mathematics than in any other subject. There are two distinct ways in which developments in technology affect learning and teaching in mathematics.

- New technology provides opportunities for new approaches to teaching and learning.
- Advances in technology impact not only on how mathematics is taught but also, on what mathematics is taught.

Modern technology also creates some new challenges and tasks in the mathematical education that must be addressed for technology to really improve teaching and learning:

- One large new challenge is to strike the right balance between work with and without technology in a blended learning approach. Some participants advocated an approach where concepts come first and their application using technology comes later.
- Learners should discuss and experience an adequate use of technology. They should for example see that there is no need for technology when it comes to computing cos(0): Students should also experience the limitations and pitfalls of mathematical technology (for example: are all possible cases covered in a computational procedure?) such that they

develop a critical attitude towards technology and can make goal-oriented use of technology.

Technology will be there and cannot be ignored.

We recognize the following new opportunities and potential advantages provided by technology:

- Better visualization of mathematical concepts by using mathematical programs.

In some area of mathematical education, the visualization and demonstration is very important for the effectiveness of the teaching process, for example, in calculus or multivariable calculus.

- Explorative approach to learning.

Using technology, the technical dimension of mathematical activities is been facilitated. It allows the user to take action on mathematical objects or representations of those objects. This feature can be utilized to enable students to explore objects and structures and to discover properties and connections e.g. by performing parameter variations.

- Experimental approach to problem solving.

Mathematics programs provide new ways of problem solving. In classical paper and pencil work students had to know a certain procedure in order to solve a problem and they could not advance once they got stuck in the process. Mathematics programs allow students to select different ways of investigating a problem (for example finding an approximate solution by looking at the graph of a function instead of getting an exact solution; investigating several related examples to derive a hypothesis or to discover a counter-examples and thus increase the student's chances in making a progress with a problem.

- Realistic modeling.

Mathematics and engineering programs allow the earlier introduction of more interesting modelling tasks since some parts of the computation can be delegated to the program.

- Change of roles.

Using mathematics programs can help to bring about changes in the way classes are conducted, as their usage requires student active participation and bigger activity. Such activity can also be designed to require interaction among students. The result is that the process of acquiring and developing mathematical knowledge becomes more student-centered. This also changes the role of teachers, who become tutors and instructors rather than lecturers.

- Large data manipulation in more realistic projects.

Experiential learning is most likely to provide expected improvements in conceptual understanding and scientific inquiry skills if teachers encourage students to repeat their experiments and provide students with a robust scaffolding to understand them.

- Easier assessment.

Real-time formative assessment (with a help of the technology) allows teachers to see in real time what students think and know, but they still have to use this information in their teaching to encourage students to reflect more deeply and to challenge their misconceptions.

- Use of chat facilities.

The use of the chat facility allowed students to participate more actively in lectures by posing questions or giving answers to questions posed by the lecturer. Why students are more comfortable with the chat option rather than asking the teacher directly is yet to be investigated. Maybe is because whereas in class only one student gives an answer, the chat allows for many students to give answers simultaneously. Moreover, the chat seemed to constitute a smaller hurdle for asking questions than the classroom situation. The question is how this can be transferred to normal class teaching. Even if the technology is available, it is hard for a lecturer to give the lecture and at the same time to observe the chat.

- Use of smartphones for activating students by voting.

Polling or voting systems are also a means to involve students more. The polling can be offered via smart phones.

- Usage of videos for flipped classroom scenarios (lecture becomes tutorial).

Many short video clips or longer recorded lectures are now available. These digital offerings increase the flexibility of learning and were hence appreciated by students.

- Motivational aspects.

Most students are accustomed to using technology such as smart phones in their daily life; consequently, simply having technology involved can make a huge difference in students' attitudes and feelings towards mathematics. Therefore, increased use of mathematics technology may help to improve student motivation for learning.

For sure, the use of technology in the mathematical education has its cons. The following risks have been identified:

- Loss of basic capabilities.

When adapting the mathematical educational process to make use of new technological tools one must be aware of the risk that this computer-based learning environment may cause an unexpected reduction of students' knowledge in basic "mathematical culture". This is not just a loss of fluency in carrying out procedural mathematical tasks brought about by a reduced amount of practice (due to using computer programs to carry out these tasks), but can also be a more limited understanding of core mathematical concepts as the reduced practice may bring with it reduced need to think about the basic concepts thereby impacting on overall mathematical reasoning skills.

- Loss of connection between procedures and understanding.

Extensive and exclusive usage of mathematical technology can potentially prevent students from making proper connections between the techniques used for calculations and conceptual understanding, for example the Gauss algorithm for solving a linear system of equations also provides insight into the possible solution types; simple use of "solve" command does not give this insight.

- Pure trial and error working style without thinking.

There is a danger that students may use mathematics and application programs in a largely thoughtless trial and error mode, making variations without any particular strategy in the hope that somehow, they will achieve what is required without having any idea why what they are doing. Problems must be found where such a strategy is not productive so that students are forced to think about the effects of possible variations.

- Tool dependence.

When students are no longer able to compute even simple examples by hand, they depend totally on what the tool they are using provides. They also have no idea of what to do when a program fails to give them an answer to a problem because they do not know what the program is able to do. Although the students do not need to know in detail what a program does, they should know which model a program is appropriate to use or not appropriate.

3. UNDERGRADUATE MATHEMATICS EDUCATION

What should Mathematics education look like now and in the future? Here we will present several innovative models of technology-supported mathematical teaching and learning. These are educational gaming, mathematical laboratories, and collaboration through technology. The emphasis is on practices that would be difficult to implement without technology and that can improve not just traditional learning outcomes, but also motivation, social, behavioral, thinking and creativity skills and their assessment.

3.1 **Educational gaming.**

Educational gaming offers a promising model to enhance student learning in mathematical education, not just improving content knowledge, but also motivation and thinking and creativity skills, see [2,3]. Educators and policy makers should consider using it to enhance mathematical learning outcomes and problem-solving skills and motivation. Designing games appears to lead to even deeper learning than just using them for educational purposes. In educational gaming students interact with video games, simulations or virtual worlds based on imaginary or real worlds, also seen as highly interactive virtual environments. Educational gaming also includes collaborative project-based learning experiences where students themselves become game designers and content producers. As a promising model for various disciplines and education levels, educational gaming may promote:

- Learning by doing.

The interactive, reactive and often collaborative nature of educational gaming enable learning by doing of complex topics by allowing students to (repeatedly) make mistakes and learn from them.

- Real-life based gaming.

Allows experimentation that would otherwise be too costly or dangerous.

- Student learning.

Educational gaming which covers specific topics or subject areas and take place within a set of rules can increase students' achievements and subject specific knowledge. Constructing educational games seems to increase deep learning more than just using existing games.

- Student engagement and motivation.

Based on play and increasing challenges, educational gaming can foster student engagement and motivation in various subjects and education levels. Low-achieving students may find the educational gaming experience more engaging than high achieving students. Students' motivation can increase more when they construct games themselves as opposed to just playing an existing game.

- Students' thinking skills.

Games have the potential to help students find new ways around challenges, use knowledge in new ways and "think like a professional". Educational gaming may also improve students' skills such as problem solving.

3.2 **Mathematics Laboratories.**

What is mathematics laboratory? Can we speak these days about Mathematics Laboratories, or maybe the Mathematics Laboratories existed some years ago, see [4]. By mathematics laboratories, we mean learning scenarios where students work in a PC laboratory on tasks requiring the use of mathematical software such as numerical programs Matlab, Maple, Mathematica or spreadsheets Excel. In such laboratory sessions, students practice the usage of the programs and see how they can be used for standard tasks. They might also be used for experimenting with more open tasks of an investigative nature.

Mathematics laboratories, whether remote or virtual, are another promising innovation to enhance the teaching and learning of mathematics at all levels of education. Virtual mathematics laboratories allow students to simulate experiments while remote ones allow students to use real laboratory equipment from a distance through the Internet.

Educators and policy makers should consider mathematics laboratories as a promising way to increase access to a wide range of experimental learning. The use of mathematics laboratories only requires access to the Internet and allows teachers and students to get access to more experimental equipment than a single school can generally provide, see [8].

3.3. **Collaboration through technology.**

Collaboration through technology can enhance students' interaction, engagement, learning and thinking skills, in addition to increasing flexibility and diversity in educational experience, see [9]. Technology-supported collaboration can enhance students' awareness of global challenges and develop their understanding of other cultures. In technology-enabled collaboration, students work together (in groups) and/or interact with each other to enhance their learning with the help of various technologies and often with facilitation from the teacher.

When combined with other learning approaches, technology-enabled collaboration can form a part of projector problem-based learning or supplement face-to-face learning. Technology-enabled collaboration models may include in-built assessment features taking into account also team performance and/or collaborative activity.

As a promising model for mathematics education and other disciplines at various education levels, collaboration through technology may improve:

- Flexibility.

Technology enables students to collaborate and practice at "their own pace", beyond the formal classroom hours and without limitations of physical location.

- Cultural diversity.

Technology can significantly increase possibilities for intercultural interactions by broadening the scope of collaborations to distant locations, even across borders.

- Student learning.

Technology-enabled collaboration may support student learning, in both individual and group outcomes, although not necessarily more than face-to-face interaction. There can also be cross-cultural differences. In general, positive results of co-operative learning on student achievement have been shown to depend on group learning goals and individual accountability.

- Student interaction and engagement.

Technology-enabled collaboration can encourage student group work skills, interaction and engagement. Yet, "active learning strategies" are not automatically adopted and activity may differ across cultures. In general, cooperative learning has shown clearly beneficial results on affective student outcomes.

- Students' thinking skills.

Online collaboration may enhance higher order thinking even more than face-to-face collaboration through "more complex, and more cognitively challenging discussions". This can also be the case for "questioning behaviors" and "project performance".

4. RESULTS OF ANALYSIS AND EFFECTS OF PILOT LECTURES AT UNS

4.1 The preparation of Piloting lectures

The preparation for piloting lectures were organized within the IO2 and IO3 in the frame of the "Guides for STEAM learning in blended environment for teachers" (the results prepared in IO1, Fig 1). The piloting teaching and learning process of STEAM contents in STEAM sense by using Bloom's taxonomy was organized as follows:

1. The problem was posed in a way that the students can remember (recognize and recall facts) and understand what the facts means;
2. The plan details appropriate for the posed problem and expected solutions were prepared;
3. The implementation was organized in a way that students can apply the facts, rules, concepts and ideas;
4. The analysis was organized in a way that students can break down the information;
5. The students were helped by teachers to create solution, to judge the values of information, to combine the parts and to make new whole.

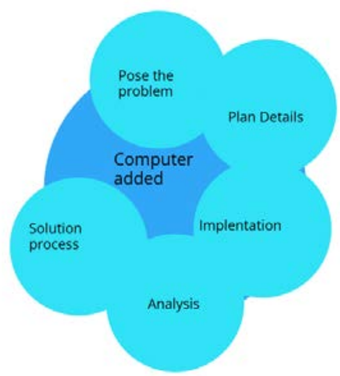


Figure 1.

The proposals for implementing STEAM technologies and methodologies prepared within IO2 were used in designing a STEAM lesson plan, teachers must consider the main idea of STEAM approaches – connection with real life problems.

- The essential thing that emphasizes STEAM lessons is presenting a real-life problem, or some engineering sciences challenge.

- Next important thing is that students have to be interested to that problem, thus it should be related to their future carrier.
- STEAM lessons make student – centered, thus students have the main role in solving given problem.
- The lesson integrates science, art and math in solving the problem.
- The lesson stimulates students' creativity in solving the problem.
- The lessons point the role of technology in solving the problem.
- Students collaborated, communicated, exchanged ideas and present their results.
- Students considered the eventual failure as a natural chain in this procedure, and as a step toward designing the successful solution.
- Students are able to design more similar real-life problems.

At the University of Novi Sad there were 8 piloting lessons;

1. **Partial derivatives**, conducted by Mirjana Brdar, face to face.
2. **Applications of the derivatives**, Max-min problems, conducted by Mirjana Brdar, face to face.
3. **Complex Functions and SageMath**, Dragan Mašulović, face to face.
4. **Solving equations in SageMath**, Dragan Mašulović, on line, film Complex numbers - Lecture.mp4 - OneDrive (live.com).
5. **Combinatorics: Splitting the numbers into sum, variations with repetitions, permutations**, Mirjana Mikalački, on line.
6. **Combinatorics: Variations without repetitions, combinations with and without repetitions** Mirjana Mikalacki, on line.
7. **Directional derivatives**, Aleksandar Takači, face to face.
8. **Applications of directional derivatives**, Aleksandar Takači, face to face.

At the Universities in Serbia there is no course called Calculus. But the Calculus contents are included in several courses depending on University. At the Faculty of Technology the courses called Matematika I and II contain Calculus contents. At the Faculty of Science, Department of mathematics and informatics Calculus contents are included in many courses. Such contents are complex numbers and combinatorics.

All piloting lessons were STEAM lessons. The guides for STEAM lesson are prepared within IO2, in "Proposals for implementing STEAM technologies and methodologies" and are used in conducting piloting lessons.

The piloting lectures were conducted at University of Novi Sad with STEM students. They were students from Faculty of Sciences, major mathematics and informatics, including future teachers. In Serbia the future teachers for mathematics and informatics are educated at the Faculty of Sciences. Also, the piloting lectures were conducted for the technology students.

Piloting lectures took place during Covid19 crisis. Therefore, at each lesson there were not so many students during the lecture. But all students the students used teaching material and test that is posted FutureMath Website.

There were 8 piloting lessons, 3 lessons were on line, and 5 lessons were face to face. There were 323 students from UNS. There were 145 (24 of them were future teachers for mathematics) from Faculty of Sciences and 178 from Faculty of Technology students.

The evaluation of each piloting lecture was conducted after the lecture. The students got questionnaire which had been made before the piloting lectures together by all partners. The survey was analyzed and confirm during LTT in Belgrade, in March 2022. The students were tested after the each course and the test are placed on Webpage. All students were tested at the University, because it was the rule during Covid19 crisis.

The feedback was positive; the results of the tests show that the students have benefited from the scaffolding provided by the infrastructure used to deliver the lecture
The piloting lecture were conducted during Covid19 crisis under certain epidemiological restrictions, therefore the attendance list are not so high, but all students were able to reach prepared teaching material.

The blended learning was applied in that time at the University of Novi Sad. Three lectures were conducted only on line:

1. Combinatorics: Splitting the numbers into sum, variations with repetitions, permutations, Mirjana Mikalački
2. Combinatorics: Variations without repetitions, combinations with and without repetitions Mirjana Mikalački,
3. Solving equations in SageMath, Dragan Mašulović, on line, film Complex numbers - Lecture.mp4 - OneDrive (live.com).

while the other five were conducted face to face with the help of computer.

4.2 STEAM methodology UNS

Steam methodology in general was scrutinized with the result of IO2 within “Proposals for implementing STEAM technologies and methodologies”. In conducted piloting lectures (8 lectures)

1. The collaborative learning was applied in 5 lectures;
2. The problem based learning was applied in 6 lectures;
3. Think/Pair/Share was applied in 2 lectures:
4. The discussion questions was applied in 8 lectures;
5. The observations was applied in 8 lectures;
6. The conversations was applied in 8 lectures;

7. The work sample was applied in 5 lectures;
8. The self assessment as learning was applied in 8 lectures;
9. The assessment of learning Test was applied in 8 lecture.

Piloting lectures

The Piloting lectures

- **Complex Functions and SageMath,** was conducted in March 23, 2022,
- **Solving equations in SageMath**, was conducted in April 20, 2022, [FutureMath - Lecture PMF 23 03 2022 - OneDrive (live.com)]

at the University of Novi Sad, Faculty of Sciences by the teacher Dragan Mašulović.

The aim of the lectures was to make students able to use SageMath to visualize complex functions and transformations and to convey the idea of number fields (Q, R, C as well as the finite ones Zp where p is prime).
As a practical problems the lecturer poses several questions that correlate finite fields to problems in cryptography and demonstrates that such problems can be modeled using finite fields.
There were 78 students in this course “Complex Numbers and Complex Functions” who were and are able to use teaching material and [UNS-Masulovic-ComplexNumbers-TeachingMaterial.pdf]

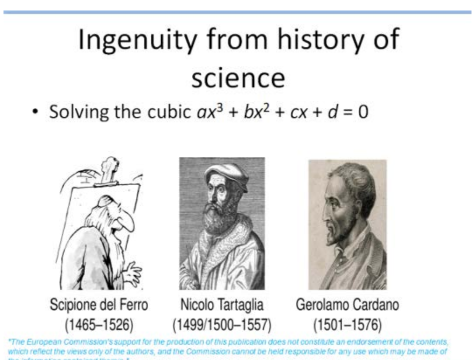


Figure 2.

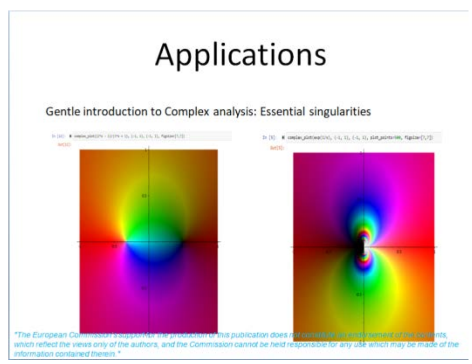


Figure 3.

Dalje… Fraktali

- Quaternions
- Further applications to Computer graphics (3D rotation!)

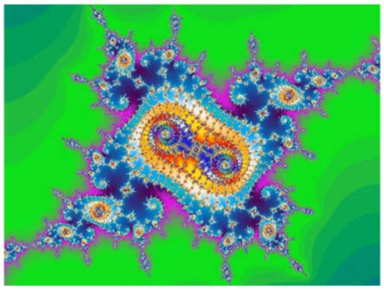


Figure 4.

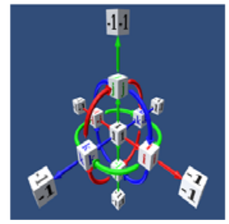


Figure 5.

In Figures 2, 3, 4, 5, the examples STEAM elements of piloting lectures “Complex Functions and SageMath”, and “Solving equations in SageMath” are presented. In Figures 4, 5, the examples of STEAM projects for students further work.

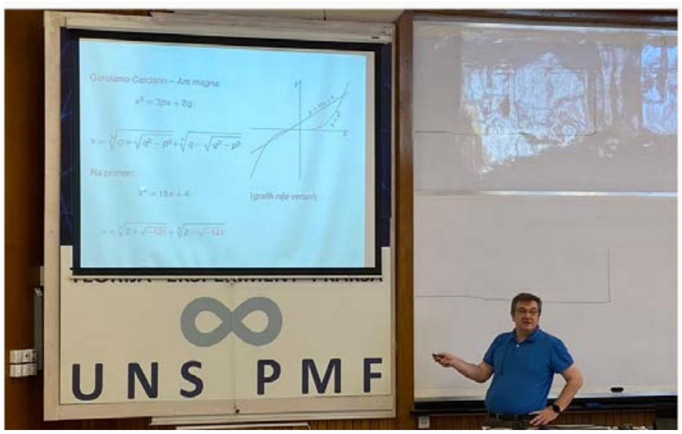


Figure 6.

Figure 6 shows professor Mašulović face to face lecture.

The Piloting lecture

- Combinatorics: Splitting the numbers into sum, variations with repetitions, permutations**,** was conducted in May 4, 2022
- Combinatorics: Variations without repetitions, combinations with and without repetitions, was conducted in May 11, 2022

at the University of Novi Sad, Faculty of Sciences by Mirjana Mikalački.

The aim of the lecture is to make students able to use Python in solving combinatorial problem, with visual solutions.
As a practical problem, the lecturer poses several questions related to the applications of combinatorial methods in real life situations.
There were 67 students, 24 of them are mathematics future teachers, in this course "Programming 1" who were and are able to use teaching material and that is available online on moodle platform.

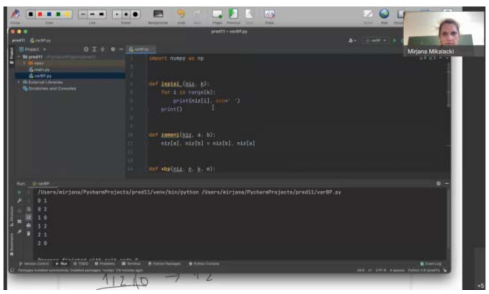
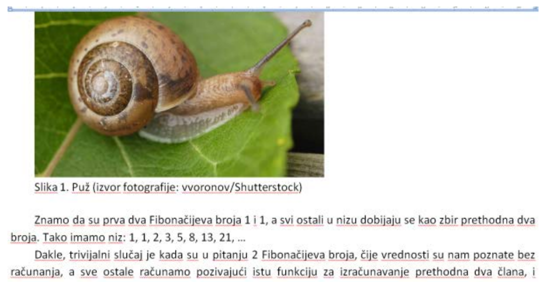


Figure 8.



Slika 1. Puž (izvor fotografije: vvoronov/Shutterstock)

Znamo da su prva dva Fibonačijeva broja 1 i 1, a svi ostali u nizu dobijaju se kao zbir prethodna dva broja. Tako imamo niz: 1, 1, 2, 3, 5, 8, 13, 21, ...

Dakle, trivijalni slučaj je kada su u pitanju 2 Fibonačijeva broja, čije vrednosti su nam poznate bez računanja, a sve ostale računamo pozivajući istu funkciju za izračunavanje prethodna dva člana, i

Figure 9.

Figure 10.

Figure 11.

Figure 8 shows the screen of online lecture of Mirjana Mikalački. Figures 9, 10 and 11 show STEAM parts of the lectures.

The Piloting lecture:

- **Directional derivatives**, was conducted in April 12, 2022
- **Gradient vector and applications of directional derivatives,** was conducted in May 10, 2022

at the University of Novi Sad, Faculty of Technology by Aleksandar Takači.

The aim of the lecture **Directional derivatives** is to adopt the notion of the directional derivative and understand its application.
Practical problem of applying directional derivatives in the problem of finding the intensity of level change on different surfaces.
In the lecture Gradient vector and application of directional derivatives the notion of the gradient vector is explained together with its use in surface descent.
The problem of largest descnt on a surface in presented and solved in detail.
There were 178   students in this course "**Matematika 2**" who were and are able to use teaching material

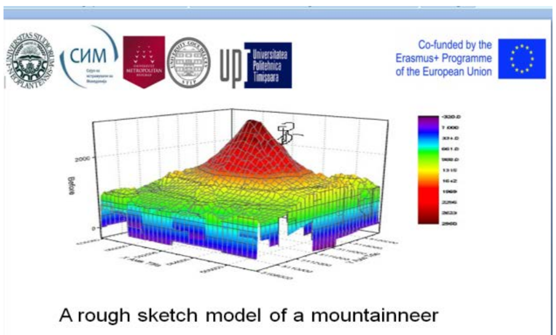


Figure 12.

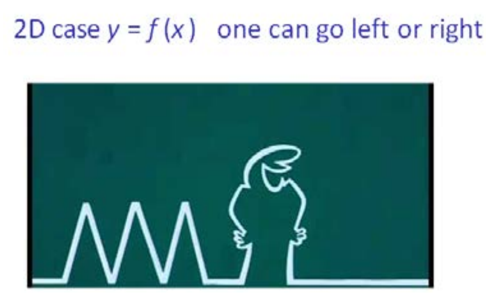


Figure 13.

Figures 12 and 13 shows STEAM approach to directional derivative.

The aim of the lecture **Gradient vector and applications of directional derivatives** is to make students able to calculate directional derivatives, gradient.
As a practical problem, the lecturer asks several questions that encourage students to understand the notion of directional derivatives.

	
Figure 14	Figure 15

The Figures 14 and 15 show the pictures of the lectures.

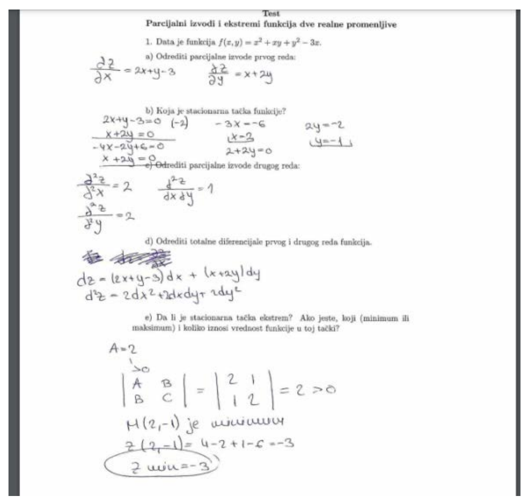
The Piloting lecture

- Partial derivatives**,** was conducted in April 13, 2022
- Applications of the derivatives, Max-min problems, was conducted in May 4, 2022

at the University of Novi Sad, Faculty of Technology by Mirjana Brdar.

The aim of the lecture Partial derivatives is to make students able to calculate partial derivatives of a function and apply the derivatives to calculate approximation of functions.
As a practical problem, the lecturer asks several questions that connect the calculation of partial derivatives and their use in real life problems.
There were 187 students in this course “Matematika 2” who were and are able to use teaching material.
The aim of the lecture **Applications of the derivatives, Max-min problems** is to make students able to calculate a stationary point of function and then determine whether there is a minimum or maximum value at that point.

Figure 16.



Test

Parcijalni izvodi i ekstremi funkcija dve realne promenljive

1. Data je funkcija $f(x,y) = x^2 + xy + y^2 - 3x$.

a) Odrediti parcijalne izvode prvog reda:

$\frac{\partial z}{\partial x} = 2x+y-3 \qquad \frac{\partial z}{\partial y} = x+2y$

b) Koja je stacionarna tačka funkcije?

$2x+y-3=0 \; (-2)$, $x+2y=0$; $-4x-2y+6=0$, $x+2y=0$; $-3x=-6$, $x=2$; $2+2y=0$, $2y=-2$, $y=-1$

c) Odrediti parcijalne izvode drugog reda:

$\frac{\partial^2 z}{\partial x^2} = 2 \qquad \frac{\partial^2 z}{\partial x \partial y} = 1 \qquad \frac{\partial^2 z}{\partial y^2} = 2$

d) Odrediti totalne diferencijale prvog i drugog reda funkcija.

$dz = (2x+y-3)dx + (x+2y)dy$

$d^2z = 2dx^2 + 2dxdy + 2dy^2$

e) Da li je stacionarna tačka ekstrem? Ako jeste, koji (minimum ili maksimum) i koliko iznosi vrednost funkcije u toj tački?

$A=2>0$

$\begin{vmatrix} A & B \\ B & C \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 2 > 0$

$M(2,-1)$ je minimum

$z(2,-1) = 4-2+1-6=-3$

$z\,min = -3$

Figure 17.

The Figures 16. and 17. show the pictures of the lecture of Mirjane Brdar.

4.3. Short overview of survey

Besides the analysis on the questions from the evaluation report (following):
In the following the answers of the students of all eight lectures are summarized and analyzed.

1. The course was well organized

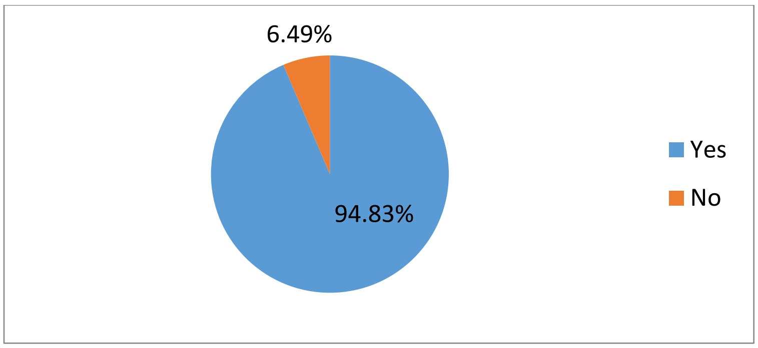


Figure 18.

One would agree that the majority of students think that the all STEAM courses, including piloting STEAM lessons were well organized, because even 94,83% (Figure 18) have opinion that the courses were well organized and only 6,49% of students thinks the opposite.
The piloting lecture were organizing and preparing from the beginning of the project FutureMath duration and therefore the whole course was claimed to be well organized.

2. Computer environment helped me to get visual approach of mathematics contents

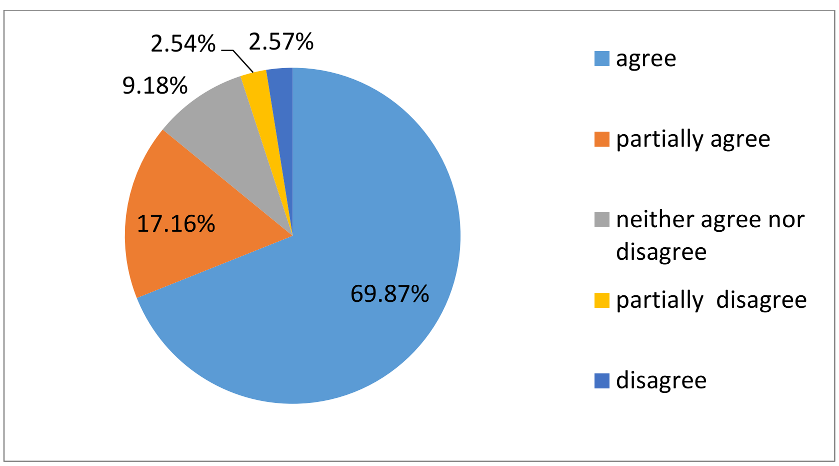


Figure 19.

The students learning environment has been change lot during the Covid19 crisis. Therefore it is necessary to get information about students opinion about that. Computer learning environment is used at University of Novi Sad, for preparing piloting lectures, for lecturing and for learning, because all teaching materials can be found on FutureMath Website. All piloting lectures were conducted with the help of computer. In particular, SAGEMAT, PYTON and GEOGEBRA educational packages were used for piloting lectures.

Most of the students, even 69,87% (Figure 19) think that the computer environment is helpful for obtaining their visual approach of mathematical contents.

3. Visualization helped me to acquire knowledge more easily

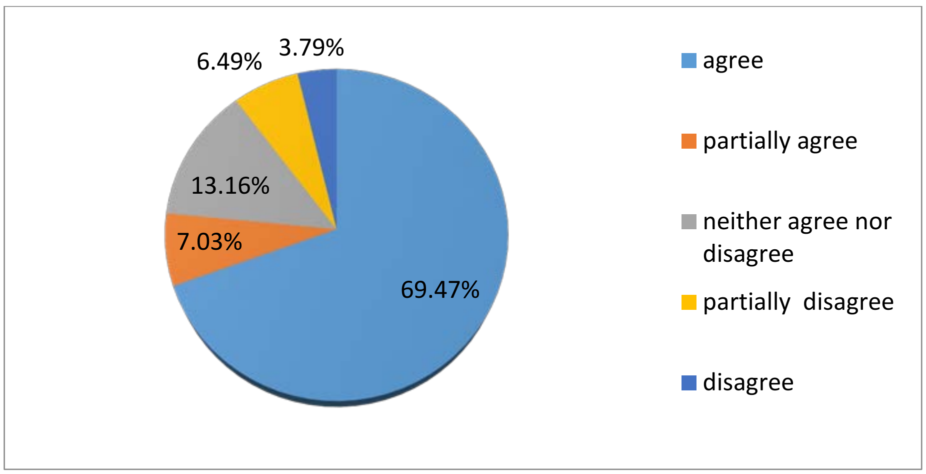


Figure 20.

The blended and on line learning was applied during the piloting lectures and educational package GeoGebra, Pyton, Sagemath, were used for the visualization of mathematics contents at the Univesity of Novi Sad.

The majority of the students, 69,47 % (Figure 20) consider that the visualization help them in acquiring mathematics knowledge. It can be considered as the confirmation of STEAM lessons because they support visualization of mathematical contents, within multiply their representations obtained by using mentioned educational packages.

4. Teaching contents are interesting

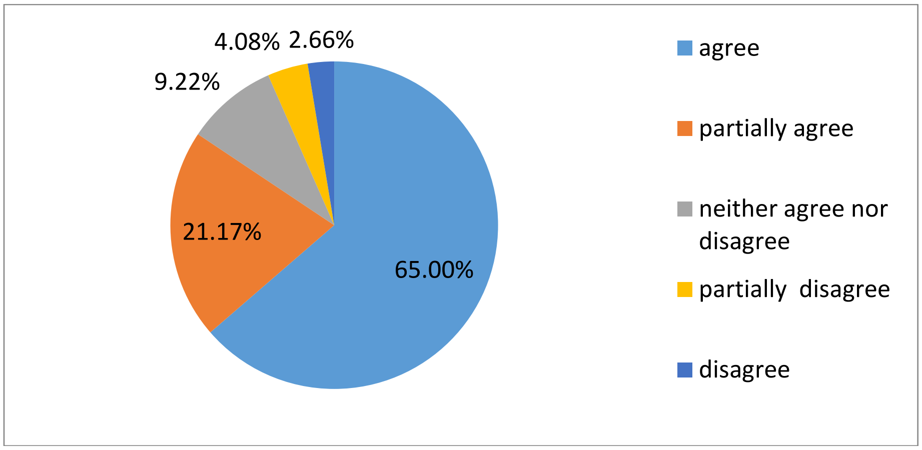


Figure 21.

Even 65% (Figure 21) of students find math piloting lectures interesting. These students opinions are valuable because it is the opposed to the public stereotype about math lectures. Since, the piloting lectures were prepped in STEAM context this means that STEAM lectures are interesting for students and they should contribute to the students' better achievement on their exams.

5. Teaching contents are applicable in everyday life

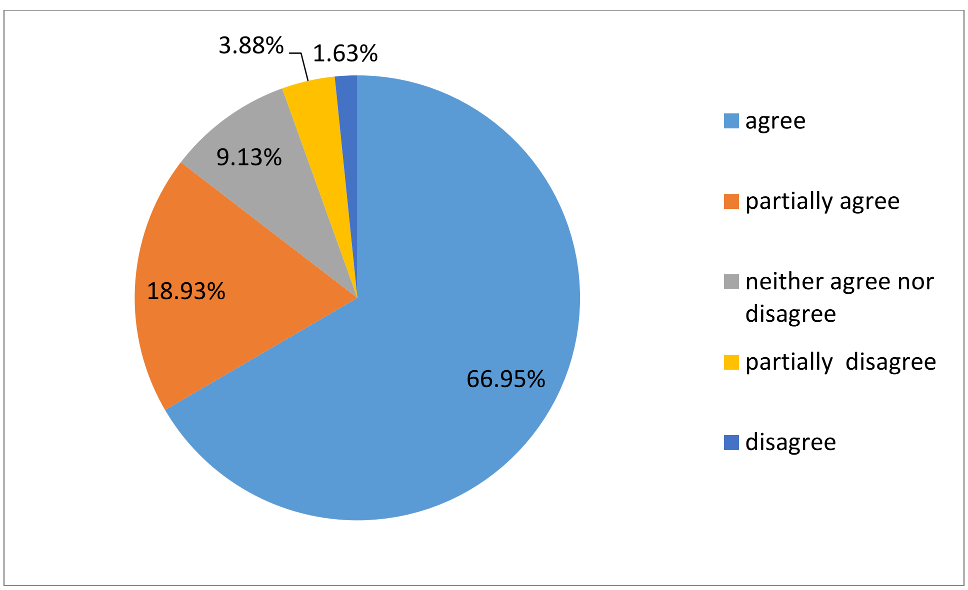


Figure 22.

Mathematical modeling process, concerning the connection between mathematics and real life problems is a part of STEAM methodology. Therefore, the fact that students, (during pilot lecturing) recognized (about 67%) (Figure 22)the application of mathematical contents in everyday life can be considered as the contribution

6. Teaching contents are applicable in sciences

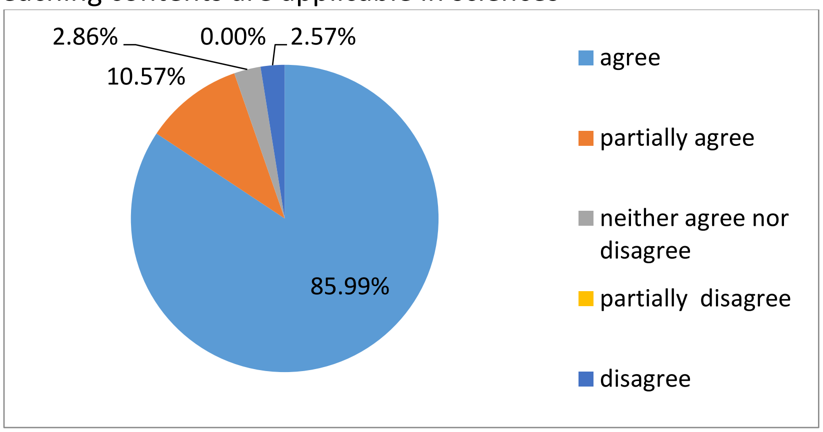


Figure 23.

This answer in expectable because we are working STEM students who understand the application of mathematics in science (around 86%) (Figure 23). Mathematical modeling of science notions is also included in STEAM methodology and therefore the students' opinion that the presented mathematics lecture are very valuable.

7. Literature is adequate for understanding the teaching contents

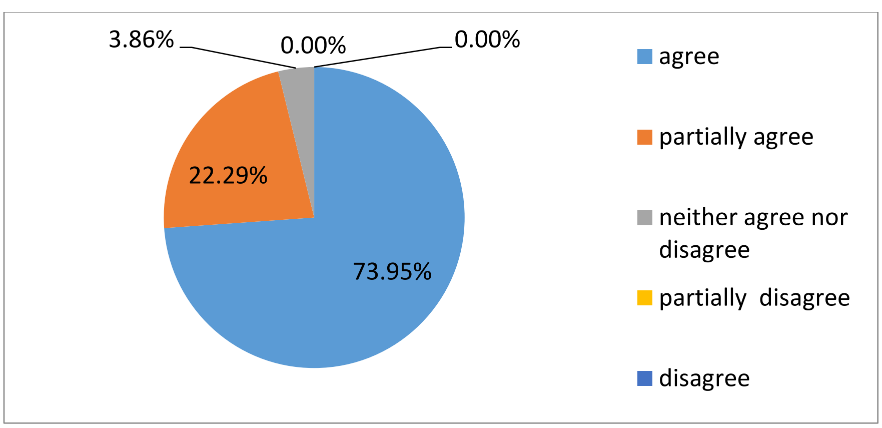


Figure 24.

As were already claimed the piloting lecture were well prepared included together with corresponding literature. Therefore the students, around 74% (Figure 24), were satisfied with the choice of literature. The literature has been chosen in order to follow STEAM concept. Therefore some math students expected more pure math literature. Nobody at UNS disagreed with the literature.

8. The communication with the teacher helped me to acquire knowledge more easily

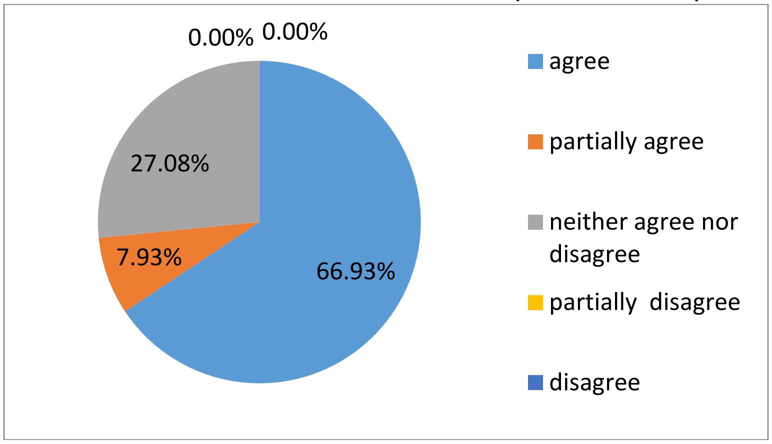


Figure 25.

The students were almost satisfied with the communication with the teacher, around 67% (Figure 25).. Nobody disagreed with communication. But there were students who had some technical problems in communication with the teacher, because the piloting lectures took place during Covid19 crisis and therefore they could not say anything about that.

9. Teaching material is available in the form: printed or electronic

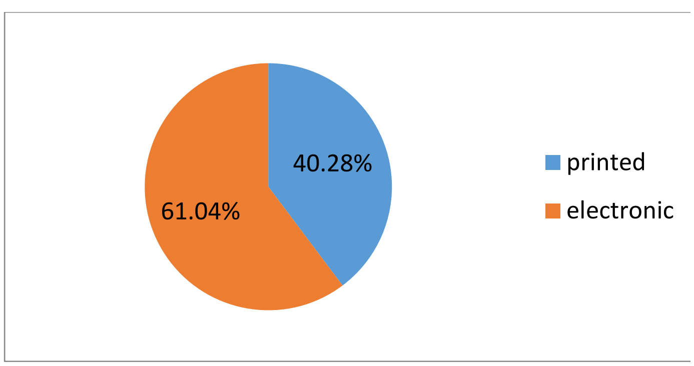


Figure 26.

This is the fact the around 60% (Figure 26) of teaching material was in electronic form and around 40% was in printed form.

In order to define the framework for integrating STEAM principles for STEM studies the report on your institution should include answers on the following questions

1. **How many answers did you get to the survey**?

We got 76 answers for all eight piloting lessons conducted at the University of Novi Sad. The teachers are explained that the students had opinion that this is not new questionnaire. They consider it as usual survey that has been conducted each year at our University.

2. **What was achieved by implementing the pilot lectures for students, teachers and your organization**?

By implementing the pilot lectures in STEAM context, **the students** obtained a different approach to problem solving, compared to the classical way of teaching. Students were first given a practical real-life problem, after which the methods for solving the problem are devised. During this process, students have built problem-solving skills and teamwork, as well as leadership skills, all of which are necessary for life outside of classroom. Even 94,83% of students from UNS, who took part in fulfilled the questionnaire are believe that the courses were well organized.

The **teachers, taking part in piloting** were well prepared for the courses. First, they were introduced with STEAM methodology, then they prepared lectures and the lectures were discussed and analyzed during LTT training in Belgrade and then the teaching material was implemented. Such well prepared teaching material is placed on project Website and it can be used by as many teachers as they are interested in teaching calculus.

The well prepared conducted and evaluated teaching material, placed on project Webpage are valuable for the students and the teachers from Faculty of Sciences, Faculty of Technology and **University of Novi Sad** at all, and for other stakeholders interested in teaching calculus and mathematics at all.

3. **How much practical knowledge does students gained from the new teaching methodology?**

This pilot lecture has benefits for the students of University of Novi Sad, of both faculty, Faculty of sciences and faculty of technology because the mathematics courses were taught with lot of practical applications in the frame of STEAM.

The STEAM methodology was applied. Collaborative learning, together with project and problem based learning and mathematical modeling contribute to students practical knowledge.

4. **Are there other important findings emerging from the implementation of the new teaching methodology developed within the project that you would like to mention.**

It is important to note that even All the students stated that the lecture was well prepared, meaning that it was interesting for them to follow. 84% of the students think that the contents are interesting, and that the communication with the teacher helped them to grasp the contents more easily.

5. **Is it possible for teachers/researchers outside the project to use the results of the project and the newly developed teaching methodology?**

The results of FutureMath project, developed proposals, guidelines, calculus courses, STEAM intervented calculus topics, are prepared in English, Serbian, Macedonian and Romanian language and placed of project Webpage. Developed teaching STEAM methodology collaborative learning, project and problem based learning as well as mathematical modeling are presented on Webpage. All of them can be used for the teachers and researches from UNS and other students and teachers interested in this teaching material.

6. **What are the major strengths of this teaching methodology?**

The major strengths of this STEAM teaching methodology is the it brings teaching to practical applications, develops problem-solving way of thinking and combines theory with practice, using all the possibilities of STEAM to promote research and creativity. The results of presented survey show that the students consider STEAM lessons interesting. (Even 65% consider piloting lectures interesting).

7. **What are the major weaknesses of teaching methodology?**

The major weakness could be the lack of computers and other technical equipment that is necessary for this type of teaching.

8. Based on the experiences of this project make recommendation on how integrate STEAM principles into STEM studies.

Since this approach was successfully implemented and was well received, the next steps include implementing other parts of the curriculum using the strategies devised for the pilot lecture. It is shown how very difficult advanced mathematics notions, as directional derivative, complex numbers, and others, can be prepared and presented visually very interesting, in a way that students can better understand them. In STEAM sense the lecture should be organized as follows:

- Starting from real life or Sciences problems, making mathematical models, working on this models mathematically connecting and analyzing in the frame of staring problems.
- Applying STEAM methodology as collaborative learning, problem and project base learning.

STEAM as a new attractive educational multidisciplinary approach , (science, new technology, engineering arts and mathematics), develop innovation, problem solving, logical thinking, and technology literacy and promote excellence. STEAM education is based on modern technology and equipment and therefore it is called the education of 21th century. Before Covid 19 crisis face to face learning had been applied in most educational institutions all over the word. Very often the educational process was improved with the help of computers, on different way. But, during Covid 19 crisis the education was changed, applying mostly distance learning, based on new technology which was rapidly penetrated in the whole education all over the world. The most of prepared teaching material was applied for piloting lessons and testing and by students and teachers opinion were helping students in overcoming difficulties in learning and understanding the various topic from Calculus.

Also, developed evaluation methods that can be used so that students can have interactive modes of learning and instant feedback when doing self-evaluation during their learning

Applying STEAM education, in particular in teaching, learning and testing students will to contribute to filling a very obvious gap in underdeveloped digital skills (e-skills) that require a modern labor market and to motivate young people to study in areas of high demand and potentially in a deep deflation with the future, which will affect the future, which will affect the future European economy as a whole.

5. Report on Implementing Piloting Lectures on Classes at Goce Delcev University

The concept of STEM education is a concept which integrates Science, Technology, Engineering and Mathematics in the process of everyday teaching and learning. Education and teaching process should not only provide students with pure knowledge, but it should answer the question why do students need that knowledge and how to apply it in the future. During the process of learning, especially in the classes, the attention of the students at each age is usually at the highest level when they are considering a real life problem and are trying to solve it. This approach, i.e. including problem-based situations, characterizes STEM education. Thus, STEM approach means introducing certain concepts and their relations in the process of teaching, as something necessary for solving different problems that student will face up in the future, in their lives or carriers.

Students usually find mathematics as difficult subject in their education and are afraid of it, which is a reason for avoiding studying engineering, technology and anything else where mathematics appears as essential. This situation can be changed, as well as students' attitude toward mathematics, if STEM approach is implemented and real life problems are introduced in the classes. STEM approach can seriously contribute in improving the perception for mathematics among students. Moreover, project-based and problem-based learning and collaboration while solving certain problems can increase communication skills, creativity and critical thinking of students. It is important a STEM approach to be implemented in the earlier education, but is very important to become everyday practice in the higher education.

In order to implement STEM approach in the process of education, teacher needs an appropriate, well-developed curriculum for their lessons. Developing STEM curriculum progressively became research interest to many teachers nowadays. In the frame of the Erasmus+ project **Mathematics of the Future: Understanding and Application of Mathematics with the help of Technology, FutureMath,** teachers who participate have developed STEM curriculum for different math topics, together with variety of teaching materials and examples how to use different digital tools, in order to easier implement STEM approach on math classes.

All developed and collected teaching materials are available as output results in the frame of the project.

Also, math teachers from the Universities that are partner institutions in this project have implemented the new approach and FUTUREMath method on lessons. They have piloted lessons mainly on calculus topics.

1. *We have to make a tin tank in a form of rectangular cuboid that will collect 125 liters liquid. Which dimensions of the tank will require the least amount of material for its construction?*

2. *We have to make box which requires 200 cm^2 cardboard for its construction. Which dimensions should the box be in order its volume to be the largest possible?*

3. *A children's playground has the shape of two circles with equal radius R=8 meters and central distance d=8 meters. In the intersection of the two circles, there are children's requisites for playing, and outside the intersection, on both sides, there is a green-grass area for playing. In order to maintain the playground, it is necessary to know the area of the green-grass section and the area of the section with children's requisites. Try to calculate both of them.*

Figure 27. Some real life problems considered on piloting lessons

In order to evaluate the successfulness of the method implementation on the piloting lessons and to compare it with the classical teaching methods, teachers have prepared short tests for checking students' knowledge and have also prepared questionnaire to evaluate students' opinion about the new introduced method.

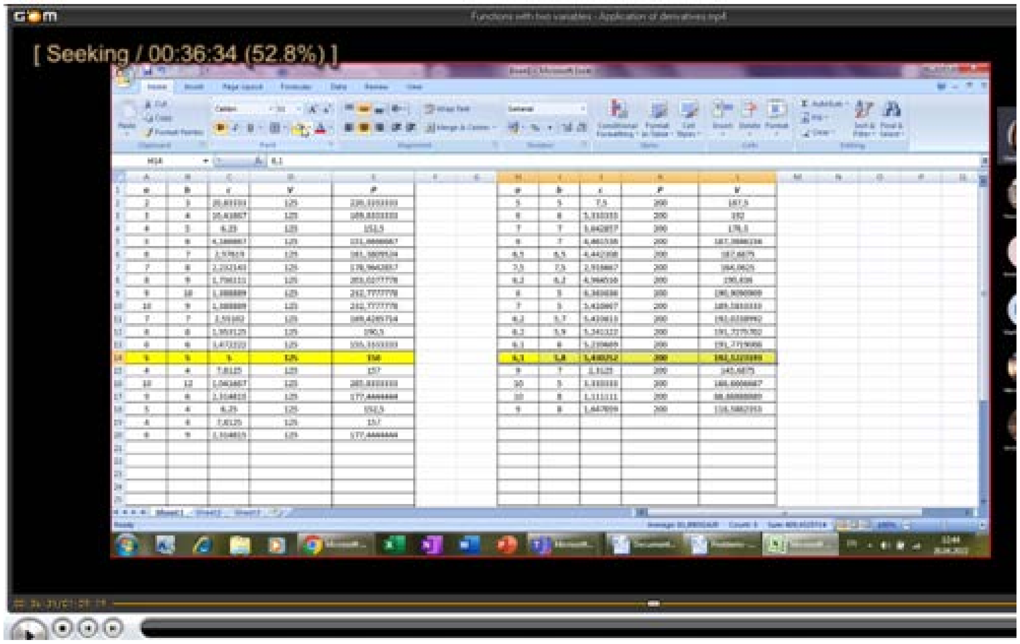


Figure 28. Using digital tools and software to help in solving problem situation (Part of piloting lecture)

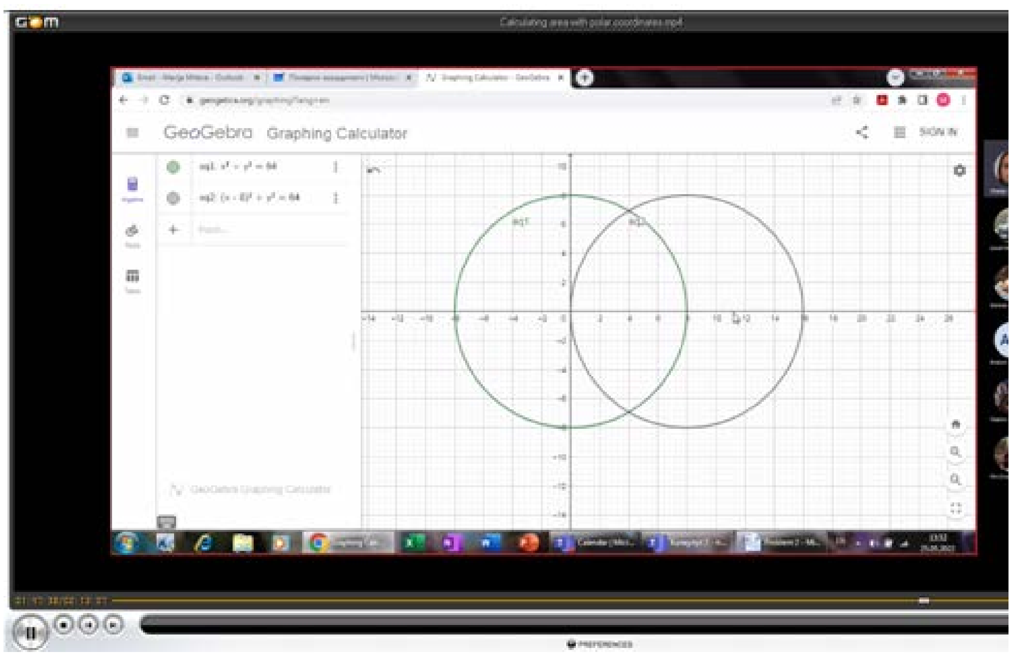


Figure 29. Using digital tools and software to help in solving problem situation (Part of piloting lecture)

Teachers from the University “Goce Delcev” – Stip have organized 6 piloting lessons during which STEM approach was introduced. The groups with the students on STEM fields at our university are about 15-25 students. About 100 students have attended those 6 lectures. Piloting lessons were organized online, according to the University’s legislative during COVID 19 period and the questionnaires were also given online, in a digital form (via Google docs). It is interesting that almost all the students have answered the questions, although the questionnaires were not obligatory. In order to achieve objective answers, e-mail addresses of the students who answered the questions were not saved. Separate questionnaire was prepared for each piloting lesson, and the results are visible for each lesson separately. Thus, GDU has about 100 answers to the 6 questionnaires.

Both students and teachers have realized benefits from the implementation of the new methodology on the piloting lessons. Mathematics is usually understood by students as a science for itself, without any connection with reality. Using old methods during the math lessons, where math concepts were introduced with classical lectures, only with black board and chalk, full with mathematical theory, formulas and expressions, students are usually passive listeners. They are usually not involved in any activity on the classes, so their attention is decreasing continuously till the end of the lessons and they are getting boring. Thus, other method which will involve students in different activities during the lecture, as problem-solving situation, collaboration with others, etc. seems to be necessary to practice on everyday lessons. By connecting the 4 components of Science, Technology, Engineering and Mathematics, STEM has interesting access for presenting and introducing new material for educational classes on academic level. Implementing STEM approach on lessons and using different digital tools in order to easily achieve knowledge is big step towards to make lectures interesting, to increase students’ attention on the lessons and contribute in reaching positive attitude among students toward mathematics. Students will become active problem solvers, they will develop creative thinking, and the most important of all they will change the perception for the process of math education, realizing that math knowledge is important for their carrier and everyday life. Therefore, STEM finds wide application in improving the educational process and general satisfaction of students and teachers.

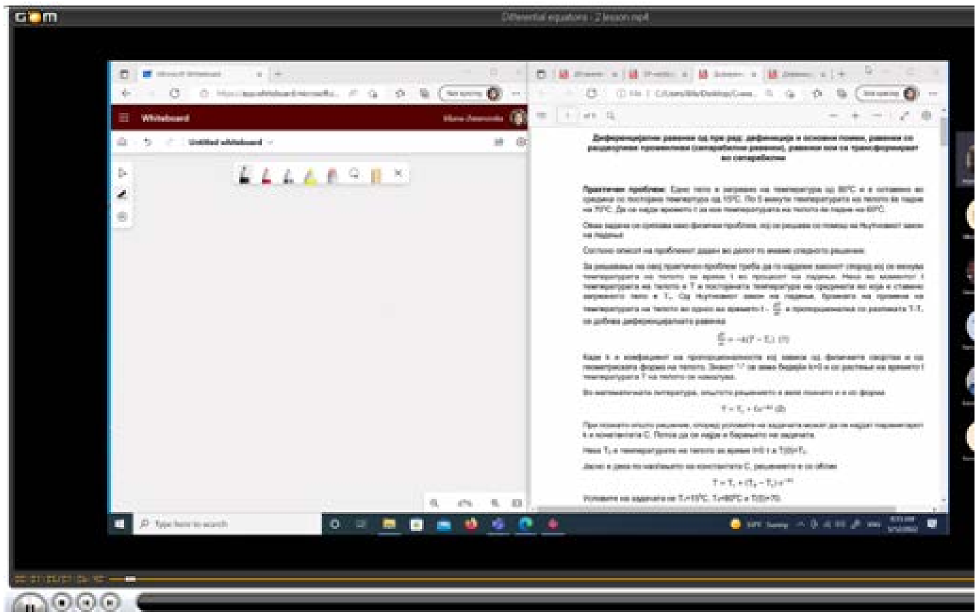


Figure 30. Segment of online piloting lecture

Implementing FUTUREmath method on piloting lessons at Goce Delcev University, the students had an opportunity to face up with these new trends of teaching and almost all above mentioned advantages of the STEM approach were achieved. By STEM as an educational approach, the students had the best introduction to each lecture via using a real problem. It is more interesting for students to consider and try to solve real life situations, than listening math lectures. The real problem motivates them to think about similar real problems, which are already known to them, without having in mind their connection with mathematics. Considering real life problems which need math knowledge to be solved, students realize that they have to achieve appropriate math knowledge first, in order to successfully solve the problem. Thus, learning math formulas and expressions become necessary, and students are not wondering anymore why they have to learn it. Presenting the new material and using computer applications and mathematical software, made the lesson more interesting and fun for the students then the previous methods of lecturing. Implementing all of this encourages the students to collaborate and discuss one with another, but also with the teacher via creative questions. These questions are related to requests for clarification of introduced new terms and curiosity to learn more, which are the basis for deeper knowledge. The lesson passed quickly, creatively, with fun, and with the mutual satisfaction of both the students and the teacher. The students passed the pilot lecture as the quality time spent because the new material was already introduced. The biggest result is satisfied students. The smiling and satisfied students, for the teacher mean successful organized lesson.

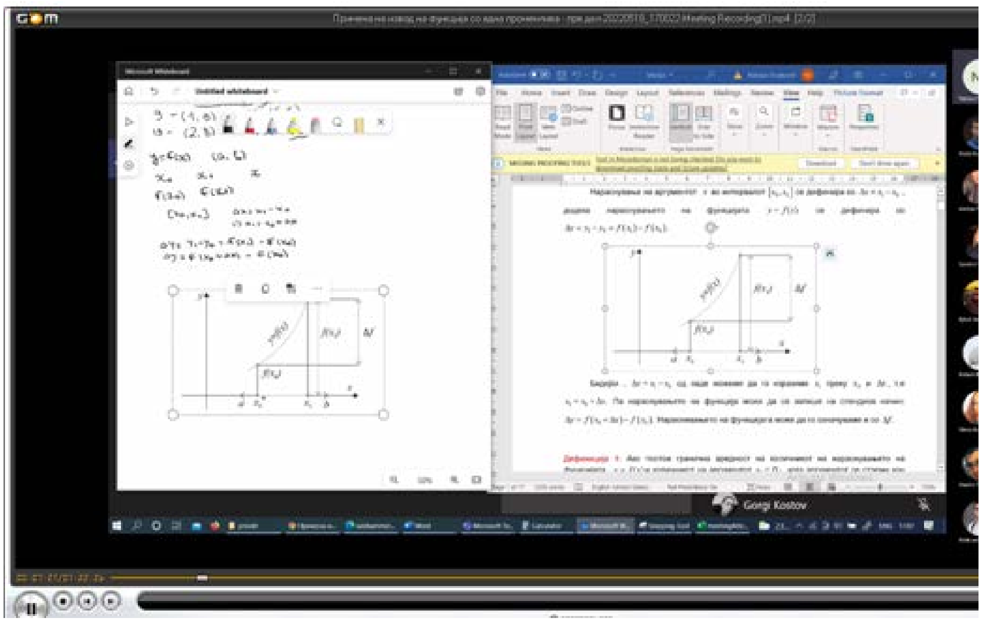


Figure 31. Segment of online piloting lecture

By these piloting lectures held, students gained the following goals:

- successful achievement of the planned material for the lesson;
- deepening of the knowledge;
- a curiosity for new mathematical challenges;
- motivation for learning;
- new creative ideas;
- quality time spent.

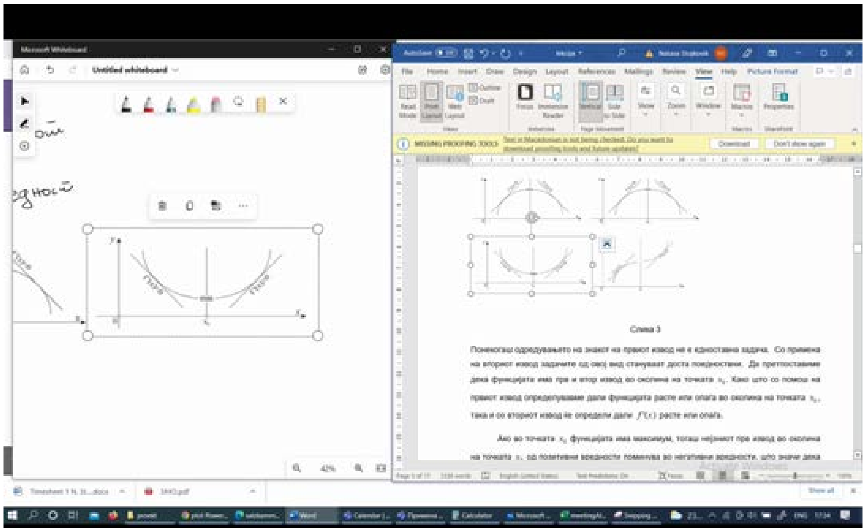


Figure 32. Segment of online piloting lecture

Increasing students' attention on the lessons and appearing interest for mathematics among students, are the biggest achievements for math teachers.

By these piloting lectures held, the teachers achieved the following goals:

- successful introduction of the material;
- successful connection of mathematics with the real world;
- encouraging motivation and creativity of the students;
- successful achievement of the material by the students in the lesson;
- satisfied students;
- successfully finishing exams and higher grades for the students.

Changing the students' perception about mathematics, as a result of the new approach in teaching, will on a long time make students to not be afraid anymore of mathematics and make students not to avoid study programs where mathematics appears as essential. Furthermore, implementing new teaching methodology in the education can attract students to study STEM fields, which will be big achievement for the Universities.

Gaining practical knowledge during the studies is very important in the educational process. The students move through the educational process and they acquire knowledge from many different (but connected) areas. Later, this knowledge they should apply in real situations in life. The initial places where students can apply theoretical knowledge in order to solve practical problems close to the real situations of their lives are the educational institutions. Educational institutions should teach the students how to do it. Only in this way, the students will be ready and successful for real life. Therefore, schools and universities must look for new ways and methodologies for practicing knowledge as a part of the process of teaching. Practical knowledge in teaching refers to students' knowledge of classroom situations and solving the practical problems they face in carrying out actions. The connection between Science, Technology, Engineering, and Mathematics guarantees the acquisition of practical knowledge in teaching. That is the STEM methodology. Having in mind that the new teaching methodology is based on solving practical problems, there is no doubt that students will gain practical knowledge simultaneously with the theoretical one. The new teaching methodology for practical knowledge per STEM means:

- connection of the real situations with the need for introducing new mathematical terms, which are provided in educational teaching programs;
- students should feel the need from introducing new mathematical knowledge;
- obtaining new knowledge with applications for solving a practical problem that is close to real problems of life;
- obtaining new knowledge which can be implemented in real situations.

The biggest benefit of gained practical knowledge is that the knowledge can be implemented in certain real-life situations.

Applying different digital tools and software while solving certain problem-based situation students also gain practical knowledge.

There are many other findings emerging from the implementation of the new teaching methodology developed within the project. By STEM methodology, the students think more broadly and more deeply than usual. The STEM methodology determines the way for the students to research new and creative ways to solve real-world problems and connect themselves to the fields that interest them. Using STEM methodology contributes in producing students who think critically with the integration of knowledge and skills from multiple areas. The students get creative ideas to apply the acquired knowledge in solving real problems. Later, these students will get up innovators, leaders, and educators of society. They will be creative people who will lead society forward in development and progress.

The newly developed teaching methodology based on the STEM approach can essentially change the way of teaching science, technology, engineering and mathematics, and more important of it, can essentially improve students results of studying and their perception about education, generally. Thus, this new approach in teaching mathematics has to be preferred for the teachers to use. The results of the project and all its outputs, involving this new developed teaching methodology will be visible and public available via the project’s web site. Lesson plans for different calculus topics, involving problem situations and mentioning which software and digital tools can be used, can be found on the project’s web site. This means that all teachers/researchers outside the project can use the results of the project and the newly developed teaching methodology.

According to all above described that characterizes the newly developed teaching methodology, FutureMath method, based on the STEM approach, no doubly there are many strengths of it. This new teaching methodology differs a lot from the classical methods of teaching. One of its major strengths is that the students are in the center of the attention. They are no more passive listeners, but they are actively involved in the activities during the lessons. This new methodology and approach based on problem solving is strengthening students’ creativity and problem-solving skills, as well as skills for collaboration and team work. The process of teaching with this new methodology has been changed from its roots. Students have new challenges on each new lesson, instead of being afraid from the material following. Teachers have also challenges to organize interesting lessons. So, the major strengths of this teaching methodology we separate are:

- connection of teaching material with real situations;
- obtaining new knowledge via solving practical problems close to life's situations;
- motivated students;

- development of creativity of the students;
- interesting lectures, which make full and interesting educational class.

Although too many advantages, there are even some weaknesses of this new approach in teaching mathematics. Not each mathematical educational class can be organized via practical problems from real life's situations. In mathematics there exist terms with abstract nature which cannot be connected to practical problems. The good organization of a class with this methodology requires full dedication to the teacher and a lot of spent time.

Also, very often students do not have available appropriate digital tools and software which can help them in solving certain problems. Even the software and digital tools are available, very often they do not have an experience in working with it. This can be considered as a weakness, but as an advantage at the same time, because students simultaneously with the new lecture can practice digital tools and similar resources.

So, the concept of STEM education and STEM principles can easily be implemented into STEM studies if teacher gives to the students a problem situation at the beginning of each lecture, in order students to realize that certain theoretical knowledge is necessary for solving such problem. In the continuation of the lesson, teacher can introduce theory of the subject, but students will realize it as something that they have to achieve in order to solve problem, not something useless and boring which is part of the subject's curriculum and that they have to memorize. In order to reach time on the lessons, teacher can give as homework to the students a problem situation related to the material which will be introduced on the next lesson. However, starting the lecture with a problem to be solved is the essential thing that makes this new approach, and changes the perception about mathematics and education at all.

5.1 The analysis of the evaluation forms

As we said at the beginning, after piloting lessons, we wanted to determine how students of the technical faculties evaluate this new, different approach of learning mathematics. In order to reach students' opinion and gladness of the new method implemented, we conduct a survey via Google forms. About each piloting lesson, separate questionnaire was prepared and given to the students immediately after the lesson, but questions were the same for each lecture.

Next, we give statistical analysis of the answers in the questionnaire, which will give us an answer on the question about successfulness of the method. Because the questions in the separate questionnaires (for each lesson) were the same, we analyze them generally, not lesson by lesson.

The survey consists of 10 questions. On the first 9 questions,

1. The class was good organized.
2. The computer applications help me in easier understanding of the mathematical content.
3. Computer applications helped me in acquiring knowledge more easily.
4. The content is interesting.
5. The teaching contents are applicable in everyday life.
6. The indicated literature is adequate for understanding the teaching content.
7. Cooperation in the class with colleagues and the teacher was successful.
8. Communication with the teacher helped me to gain knowledge more easily.
9. Learning with this approach helped me gain knowledge more easily,

students can choose one of the following choices:

I agree, I agree partially, I don't agree, I cannot say.

On the last question:

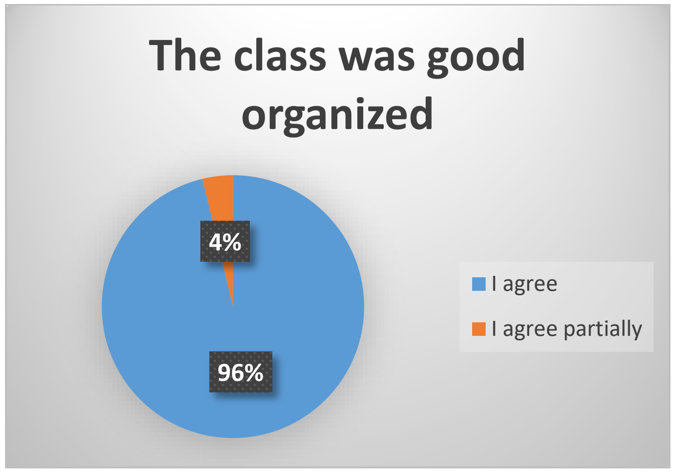
10. The teaching materials are available in the following form,

students can choose one of the following choices:

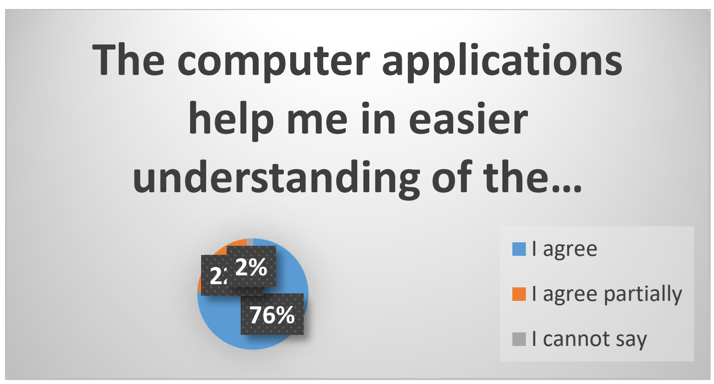
In electronic form, Printed.

The survey was filled by 107 students from the technical faculties at Goce Delcev University. The results are following:

On the first question, 103 students agreed that the class was good organized, and 4 students agree partially.

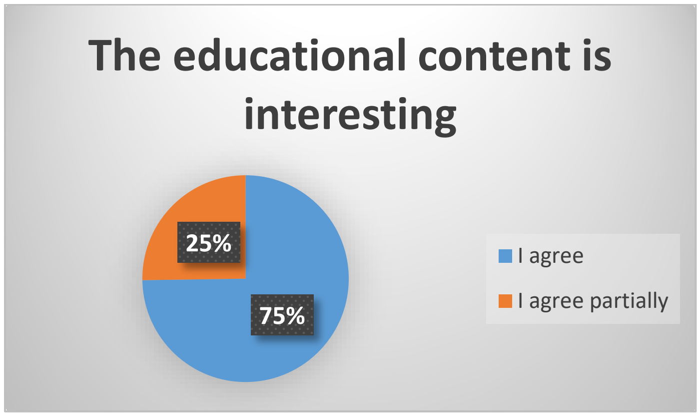


On the second question, 81 students agreed that the computer applications help them in easier understanding of the mathematical content, 24 students agree partially, and 2 students cannot say anything about that.

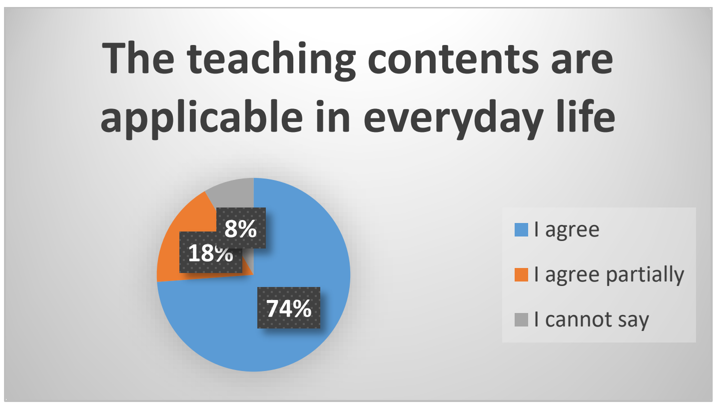


On the third question, 73 students agreed that the computer applications helped them in acquiring knowledge more easily, 32 students agree partially, and 2 students cannot say anything about that.

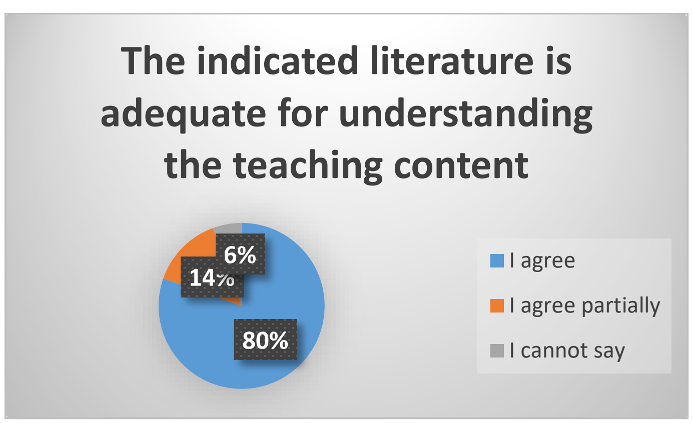
On the fourth question, 75% agreed that the content is interesting and 25% agreed partially.



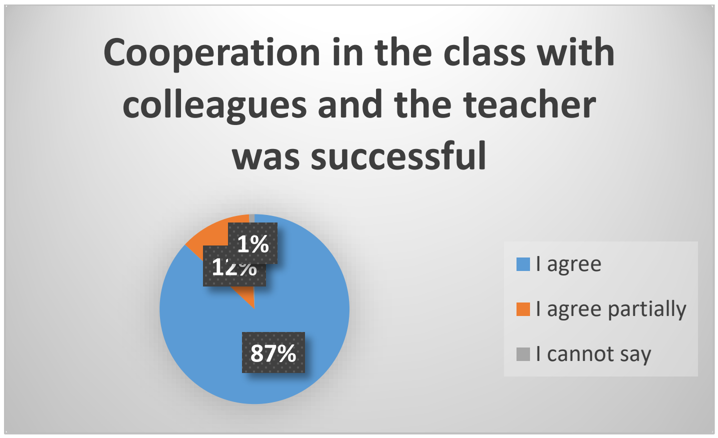
On the fifth question, that the teaching contents are applicable in everyday life, 74% have agreed, 18% have agreed partially, and 8% cannot say anything about that question.



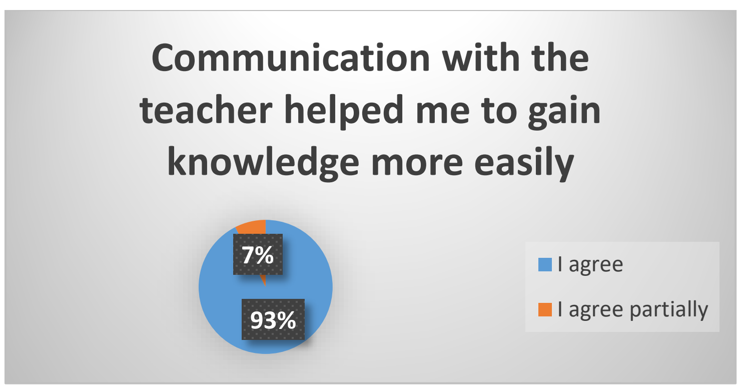
On the sixth question, the indicated literature is adequate for understanding the teaching content, 86 students have agreed, 15 students have agreed partially, and 6 students cannot say anything about that.



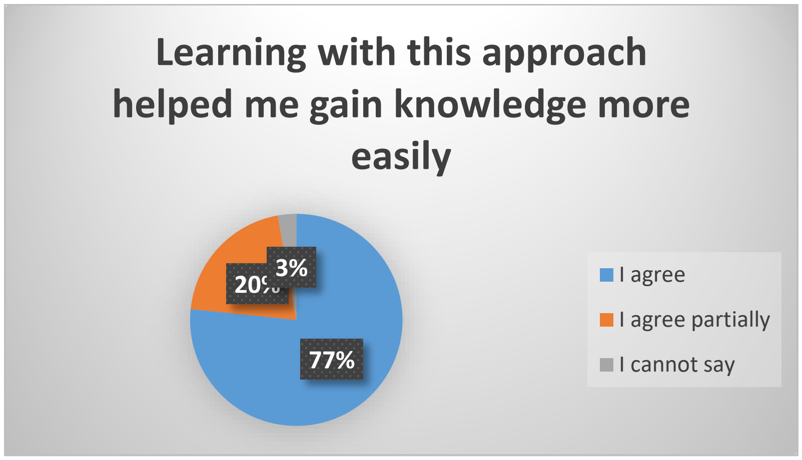
On the seventh question, about the cooperation at the classes with students and professor, 93 students agreed that there was cooperation, 13 students partially agreed and only 1 student cannot say anything.



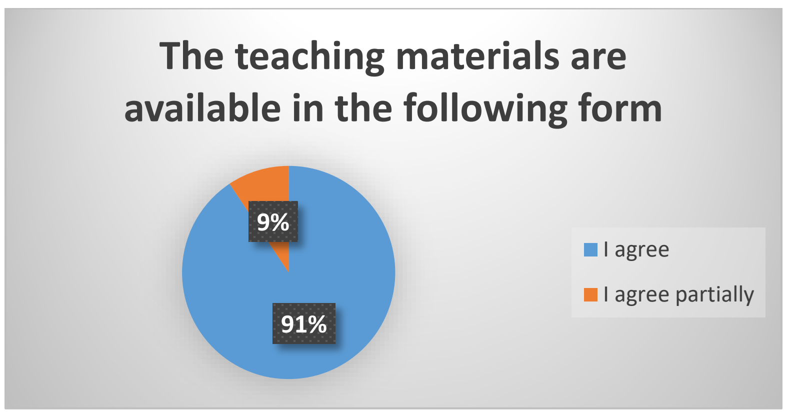
On the eighth question, about the communication with the teacher helped them to gain knowledge more easily, 99 students agreed and only 8 students agreed partially.



On the ninth question, that learning with this approach helped them to gain knowledge more easily, the results are:



On the last question from the questionnaire, about the form of the materials, the responses are:



Having in mind that students knew that the answers were anonymously, because they were not asked to put an e-mail address first and then answer, we hope that the most of the answers are objective. According to the results, we can conclude that the main goals of this new methodology were gained.

5.2 **Report on testing**

Short tests for the students in order to check their knowledge achieved on piloting lessons, were prepared by the teachers who implemented piloting lessons. Some of the tests were done online at the end of the lessons, and some were done face to face, at the end of the semester.

These tests do not have the same goal as the regular testing. The goal of these tests was to check how much students had understood the lectures. Therefore, the tests are short and elementary.

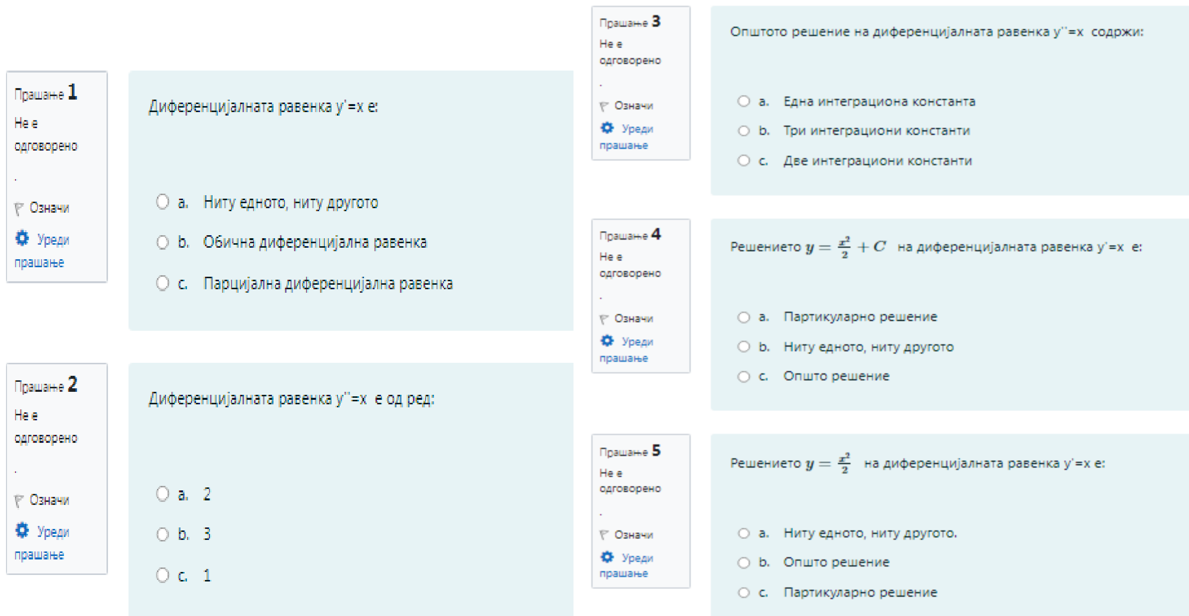


Figure 33. Part of short online test

Институција	Статус	Started on	Completed	Искористено време	Поени/20,00	Q. 1 /3,33	Q. 2 /3,33	Q. 3 /3,33	Q. 4 /3,33	Q. 5 /3,33	Q. 6 /3,33
Машински факултет	Завршен	19 May 2022 10:48	19 May 2022 10:49	1 мин 30 сек	13,33	✔ 3,33	✔ 3,33	✘ 0,00	✔ 3,33	✘ 0,00	✔ 3,33
	Завршен	19 May 2022 10:49	19 May 2022 10:51	2 мин 20 сек	13,33	✘ 0,00	✔ 3,33	✔ 3,33	✔ 3,33	✘ 0,00	✔ 3,33
	Завршен	19 May 2022 10:50	19 May 2022 10:52	1 мин 33 сек	10,00	✔ 3,33	✔ 3,33	✘ 0,00	✔ 3,33	✘ 0,00	✘ 0,00
	Завршен	19 May 2022 10:50	19 May 2022 10:51	57 сек	13,33	✔ 3,33	✔ 3,33	✘ 0,00	✔ 3,33	✘ 0,00	✔ 3,33
Факултет за природни и технички науки	Завршен	19 May 2022 10:50	19 May 2022 10:53	2 мин 21 сек	3,33	✘ 0,00	✘ 0,00	✘ 0,00	✘ 0,00	✘ 0,00	✔ 3,33
Машински факултет	Завршен	19 May 2022 10:51	19 May 2022 10:53	1 мин 47 сек	3,33	✘ 0,00	✘ 0,00	✘ 0,00	✘ 0,00	✘ 0,00	✔ 3,33
Факултет за природни и технички науки	Завршен	19 May 2022 10:52	19 May 2022 10:53	1 мин 17 сек	16,67	✔ 3,33	✔ 3,33	✔ 3,33	✔ 3,33	✘ 0,00	✔ 3,33

Figure 34. Part of the results of online testing

Институција	Статус	Started on	Completed	Искористено време	Поени/20,00	Q. 1 /3,33	Q. 2 /3,33	Q. 3 /3,33	Q. 4 /3,33	Q. 5 /3,33	Q. 6 /3,33
Машински факултет	Завршен	19 May 2022 10:45	19 May 2022 10:48	2 мин 23 сек	**16,67**	✓ 3,33	✓ 3,33	✗ 0,00	✓ 3,33	✓ 3,33	✓ 3,33
	Завршен	19 May 2022 10:46	19 May 2022 10:50	4 мин 8 сек	**0,00**	✗ 0,00	✗ 0,00	✗ 0,00	✗ 0,00	✗ 0,00	✗ 0,00
Факултет за природни и технички науки	Завршен	19 May 2022 10:46	19 May 2022 10:51	5 мин 13 сек	**20,00**	✓ 3,33	✓ 3,33	✓ 3,33	✓ 3,33	✓ 3,33	✓ 3,33
	Завршен	19 May 2022 10:47	19 May 2022 10:49	2 мин 35 сек	**10,00**	✓ 3,33	✓ 3,33	✗ 0,00	✗ 0,00	✗ 0,00	✓ 3,33
Машински факултет	Завршен	19 May 2022 10:47	19 May 2022 10:50	2 мин 47 сек	**20,00**	✓ 3,33	✓ 3,33	✓ 3,33	✓ 3,33	✓ 3,33	✓ 3,33
	Завршен	19 May 2022 10:48	19 May 2022 10:50	1 мин 26 сек	**10,00**	✓ 3,33	✗ 0,00	✓ 3,33	✗ 0,00	✗ 0,00	✓ 3,33

Figure 35. Part of the results of online testing

The results of the tests show that some of the students had understood to a big extent the material that was presented in the classes, but some of them have average results. In this way, the teacher has cognition for the level of understanding of the material from the students. Having in mind that the students face up the first time with such approach in teaching, and with testing immediately after the lesson, we find it as the reason for average results. Students were probably aware that the results will not affect their grades. The results were much better on the testing at the end of semester, because students knew that the results will have impact on their final grades.

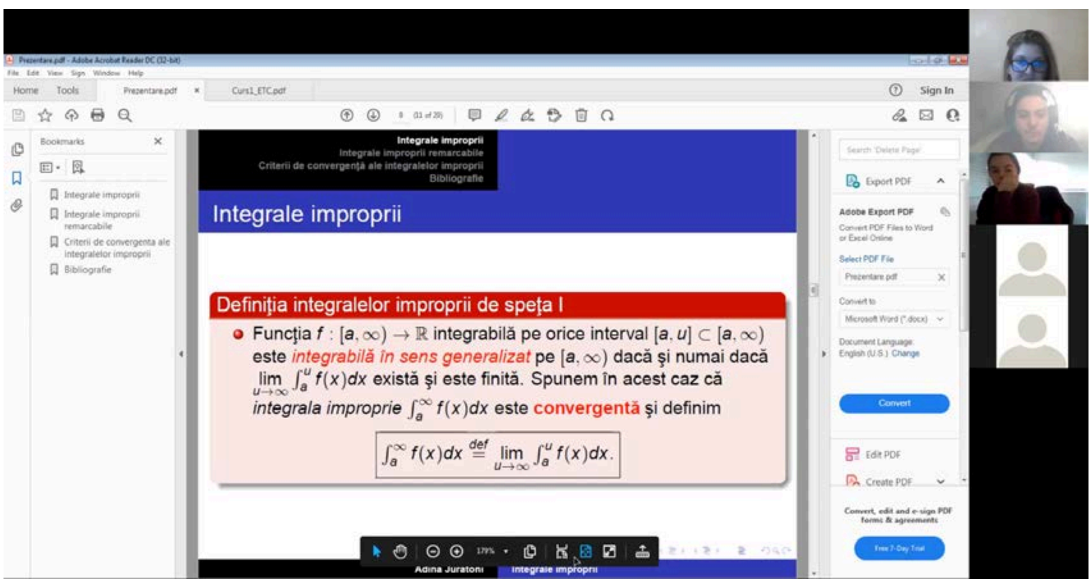
However, students’ attention and interest for the lectures was increased to a significant level while teaching with the new approach. Thus, students’ results on testing will no doubt be better if the method is applied continuously.

6. RESULTS OF ANALYSIS AND EFFECTS OF PILOT LECTURES AT UPT

One of the most important goals of the project is the development and implementation of Piloting lessons. Based on the representative lessons presented in O3 and making use of the vast amount of experience accumulated in the (almost) two years of online teaching, the teachers included in the project were able to prepare a set of Pilot lessons which made full use of the available technology. In the case of online lessons we employed video conference platforms (the official UPT platform is Zoom), document webcams/whiteboards, mathematical software (Mathematica, Geogebra). In the case of face-to-face lessons we also employed mathematical software by means of laptops and video projectors.

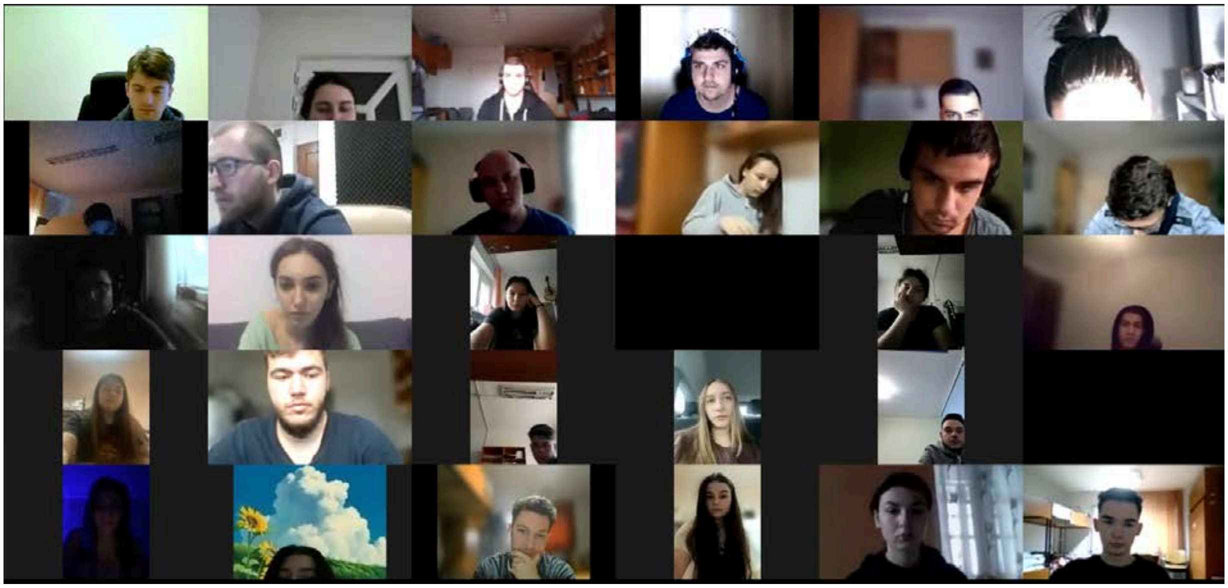
At the "Politehnica" University of Timisoara so far we included in this project five Pilot lessons. Three of them were online lessons, "Integrale improprii" and "Integrale duble" ("Improper integral" and "Double Integral", both in Romanian) for the first year students of the Faculty of Electronics, Telecommunications and Information Technologies, part of the "Analiza Matematica 2" course ("Calculus 2") and "Integrale curbilinii" ("Line Integral", in Romanian) for the first year students of the Faculty of Electrical and Power Engineering, part of the "Matematici Speciale" course ("Special Mathematics"). The other two were face-to-face lessons, "Numerical Solutions for Differential Equations" (in English) for the first year students of the Engineering in English (ICE) bachelor program at the Faculty of Civil Engineering, part of the "Mathematics 4" course and "Solutii numerice pentru ecuatii diferentiale" (in Romanian) for the first year students of the Faculty of Electrical and Power Engineering, part of the "Matematici Asistate de Calculator" course ("Computer Assisted Mathematics"):

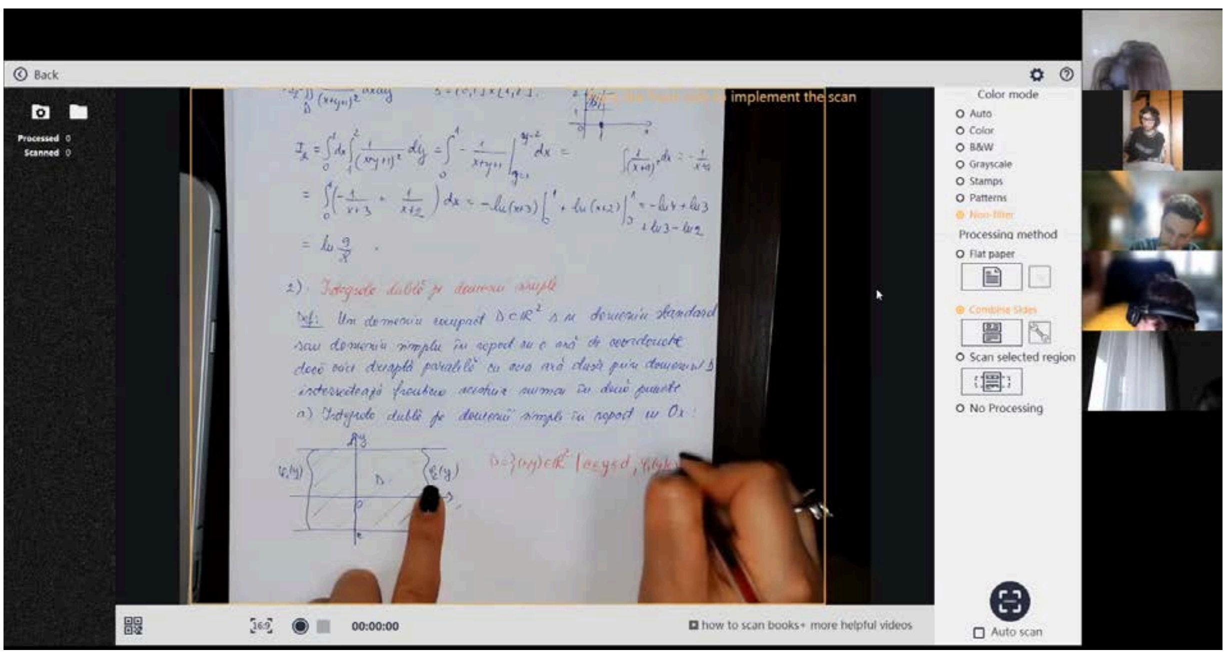
"Integrale improprii"("Improper integral", in Romanian) for the first year students of the Faculty of Electronics, held online by Prof. Adina Juratoni.



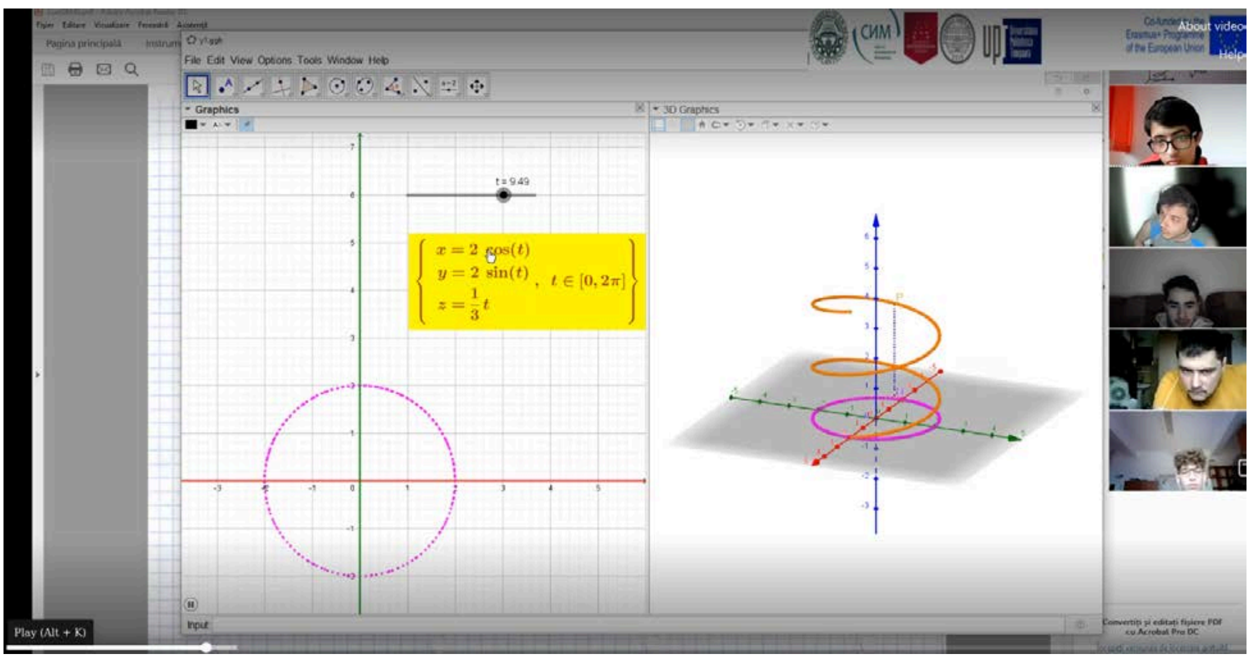
"Integrale duble" ("Double Integral", in Romanian) for the first year students of the Faculty of Electronics, held online by Prof. Adina Juratoni.

Co-funded by the
Erasmus+ Programme
of the European Union

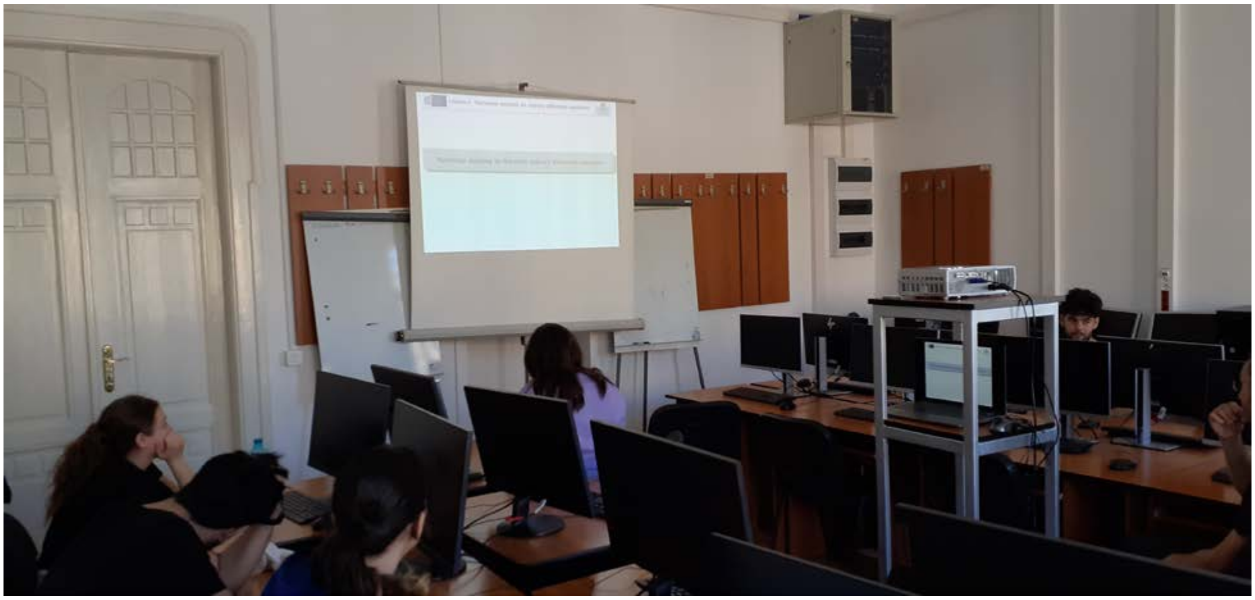


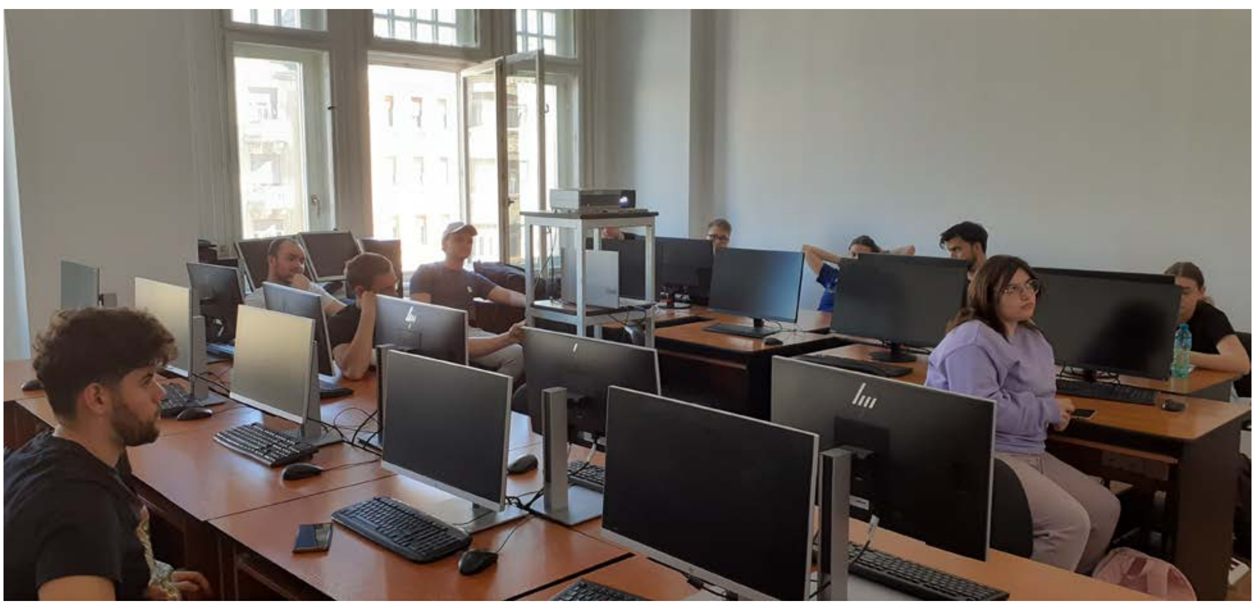


***“Integrale* curbilinii” (“Line Integral”, in Romanian) for the first year students of the Faculty of Electrical and Power Engineering, held online by Prof. Doru Paunescu.**



“Numerical Solutions for Differential Equations” (in English) for the first year student of the Engineering in English (ICE) bachelor program at the Faculty of Civil Engineering, held face-to-face by Prof. Bogdan Caruntu.





“Solutii numerice pentru ecuatii diferentiale ” (in Romanian) for the first year students of the Faculty of Electrical and Power Engineering, held face-to-face by Prof. Bogdan Caruntu:

In order to asses the results of our efforts and the outcome of the Pilot lessons, the students attending these events were asked to fulfill a short online survey containing the following entries:

1. The class was well organized
2. Computers helped me to visualize the presented content

3. Visualization helped my learning
4. Lecture content is interesting
5. Lecture content is applicable in practice
6. Lecture content is applicable in science
7. There is adequate literature that helps learning the content
8. Communication with the teacher helped me to adopt the content more easily
9. I was able to access the material: electronically / paper

With the exception of the last entry, where the choice was between electronic of paper materials, the rest of the entries had the following possible choices: “Agree”, “Partially agree”, “Neither agree nor disagree”, “Partially disagree” and “Disagree”. In the following pages we present and briefly discuss, entry by entry, the results of the surveys. We will employ the following abbreviations:

- “**Impr_Int_ETC**” - the “Integrale improprii” Lesson for the students of the Faculty of Electronics, Telecommunications and Information Technologies.

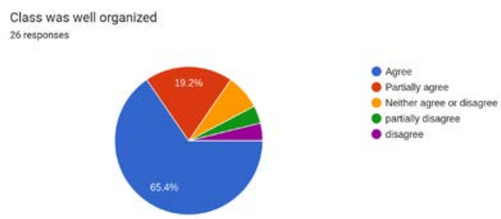
- “**Double_Int_ETC**” - the “Integrale duble”Lesson for the students of the Faculty of Electronics, Telecommunications and Information Technologies.

- “**Line_Int_ET**” - the “Integrale curbilinii” Lesson for the students of the Faculty of Electrical and Power Engineering

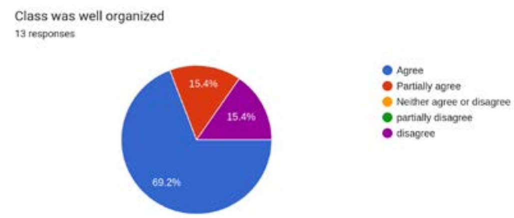
- “**Num_Sol_Diff_Eq_ICE**” for the “Numerical Solutions for Differential Equations” Lesson for the students of the Engineering in English (ICE) bachelor program at the Faculty of Civil Engineering.

- “**Sol_Num_Ec_Dif_ET**” for the “Solutii numerice pentru ecuatii diferentiale” Lesson for the students of the Faculty of Electrical and Power Engineering.

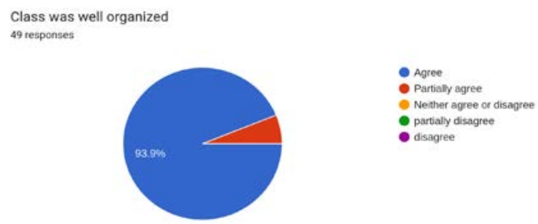
Entry 1: The class was well organized



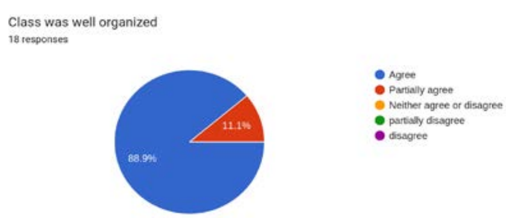
Impr_Int_ETC



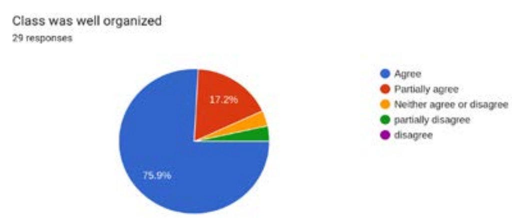
Double_Int_ETC



Line_Int_ET



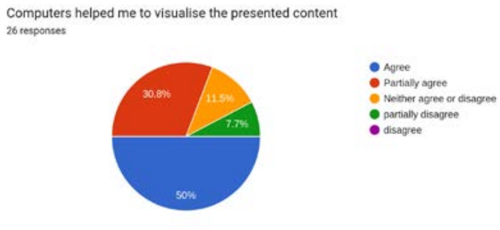
Num_Sol_Diff_Eq_ICE



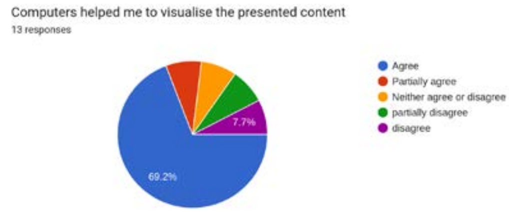
Sol_Num_Ec_Dif_ET

The results of the first entry are relatively good, most of the students considered the Lessons to be well organized. Of course, the results are influenced by the topic of the Lesson itself: Lessons including more abstract or difficult topics (such as for example *Impr_Int_ET*C) may well be perceived as less organized, especially by less prepared students – the influence of the topic itself can be perceived in all the following results.

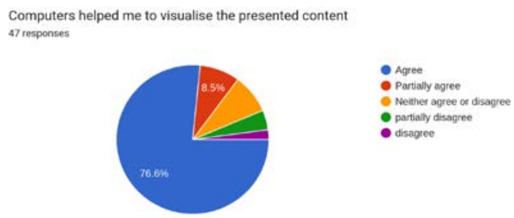
Entry 2: Computers helped me to visualize the presented content



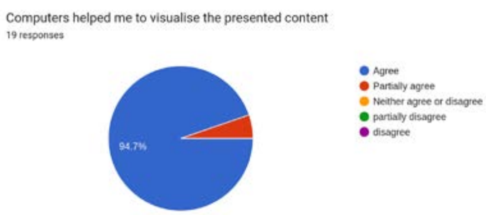
Impr_Int_ETC



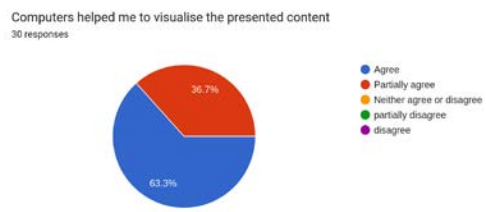
Double_Int_ETC



Line_Int_ET



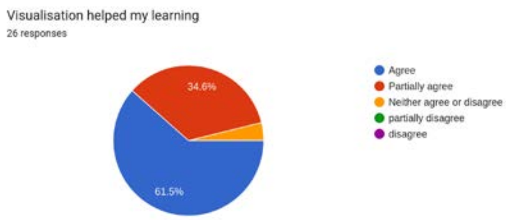
Num_Sol_Diff_Eq_ICE



Sol_Num_Ec_Dif_ET

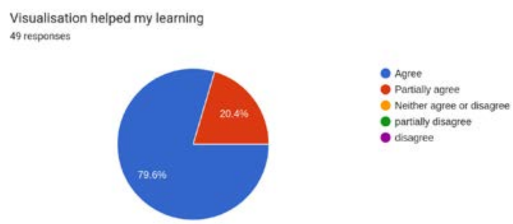
The results of the second entry are also good, most students seem to be satisfied here. Again the influence of the topic itself is visible if we compare the first three lessons related to integrals, which are usually solved by hand, with the last two lessons related to numerical solutions for differential equations, where the solution is in fact calculated using a computer.

Entry 3: Visualization helped my learning

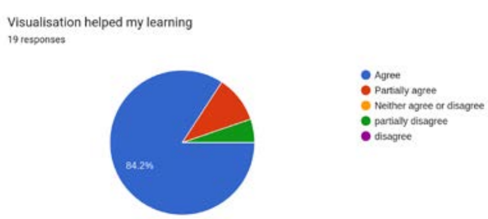


Impr_Int_ETC

Double_Int_ETC



Line_Int_ET

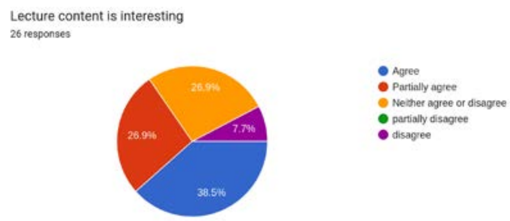


Num_Sol_Diff_Eq_ICE

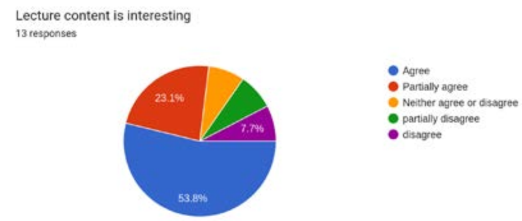
Sol_Num_Ec_Dif_ET

The results of the third entry are satisfactory. Most of the students agree (or partially agree) that the visualization was helpful, and this is an encouraging result. Still, this is a crucial aspect of the lessons which, by means of the available technology, can and must be continuously improved, especially by means of using mathematical software: Mathematica, Matlab, Geogebra etc.

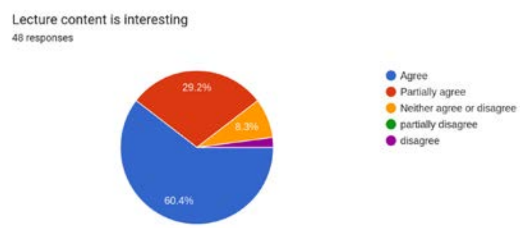
Entry 4: Lecture content is interesting



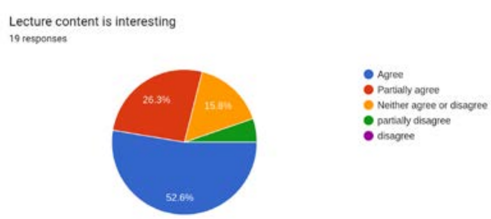
Impr_Int_ETC



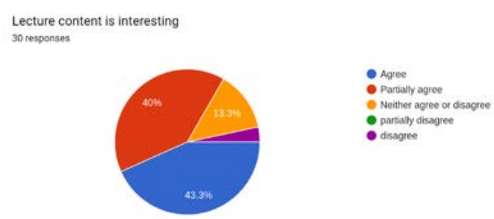
Double_Int_ETC



Line_Int_ET



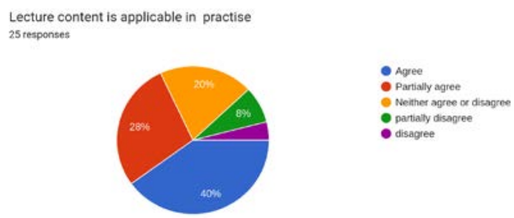
Num_Sol_Diff_Eq_ICE



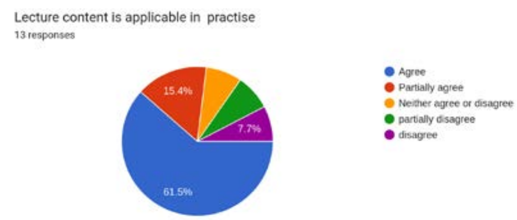
Sol_Num_Ec_Dif_ET

Here the results are not so good. These are engineering students, and unfortunately, for the case of a sizable part of them, the level of highschool mathematical knowledge is not as high as it should be. Thus the more abstract mathematical topics are a real challenge and very hard to be perceived as interesting. The presentation of the lesson an area where our efforts should be focused in order to capture the interest of the students.

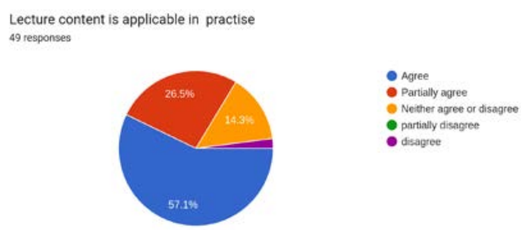
Entry 5: Lecture content is applicable in practice



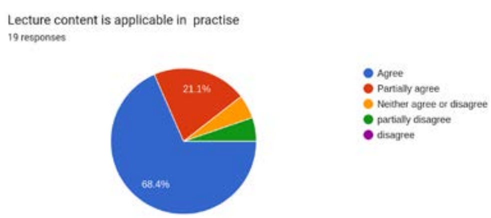
Impr_Int_ETC



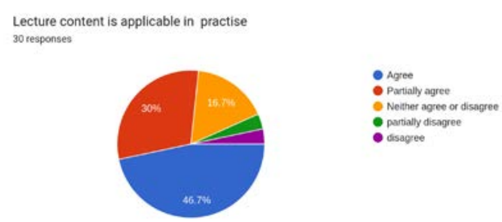
Double_Int_ETC



Line_Int_ET



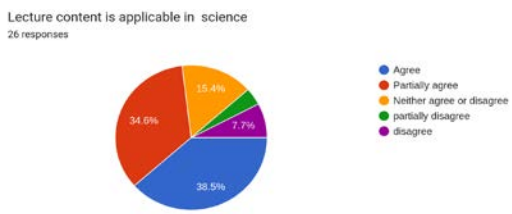
Num_Sol_Diff_Eq_ICE



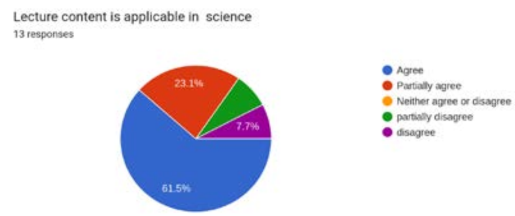
Sol_Num_Ec_Dif_ET

This sadly is another aspect which must be improved, especially in a technical university such as ours: the applicability of the content must be emphasized by relevant examples, correlated, if possible, with the domain of specialization of the students (electronics, telecommunications, power engineering, civil engineering etc.). Of course this is difficult in the case of some abstract calculus topics, but an effort should be made in this regard.

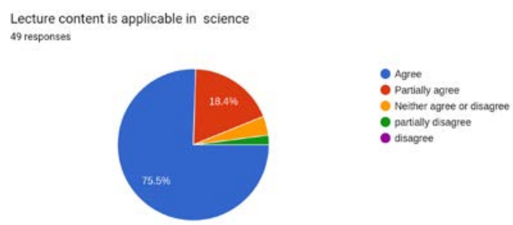
Entry 6: Lecture content is applicable in science



Impr_Int_ETC



Double_Int_ETC



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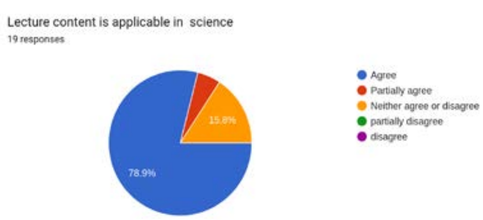
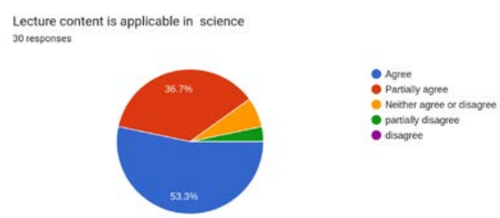


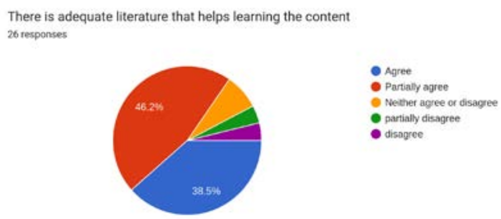
Num_Sol_Diff_Eq_ICE



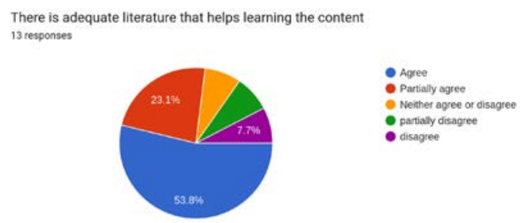
Sol_Num_Ec_Dif_ET

The results here are, unsurprisingly, similar to the previous ones. To be fair, it may be difficult for the first year students to assess the applicability of the calculus results in other areas of science since they are just starting their studies. However it is our responsibility as teachers to emphasize, if possible by examples, the applicability of mathematical results to other areas of science ("...you will use this type of integral next year in the *X* course…").

Entry 7: There is adequate literature that helps learning the content



Impr_Int_ETC



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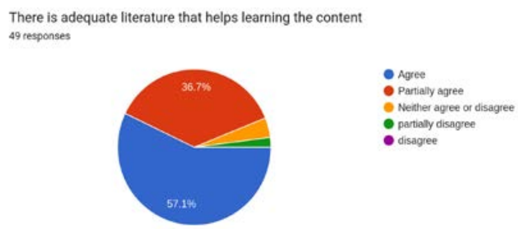

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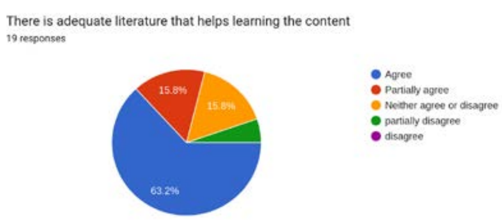
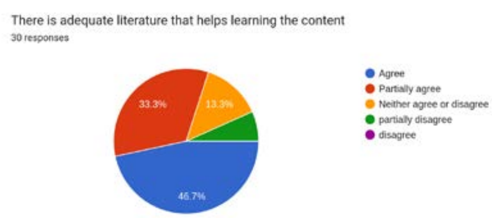


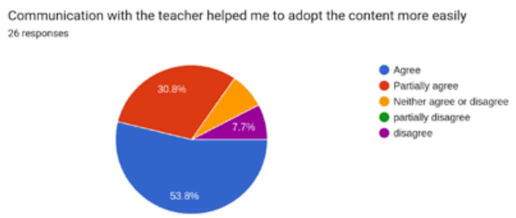
Num_Sol_Diff_Eq_ICE



Sol_Num_Ec_Dif_ET

For the courses involved here the teaching materials available online are extensive and include the content of the lessons/presentations, video recordings in the case of online lessons and proposed problems/quizzes. Also hard copies of the courses are available at the UPT Library. The results however suggest that the students may appreciate extra materials, so maybe, after consulting the students, we should add some more.

Entry 8: Communication with the teacher helped me to adopt the content more easily



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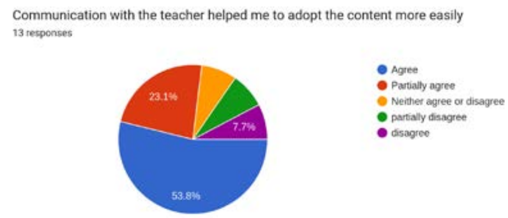

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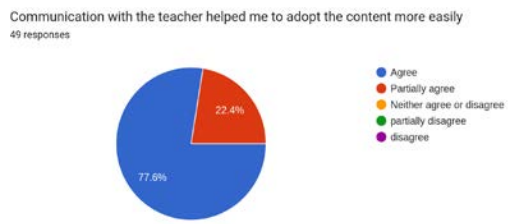

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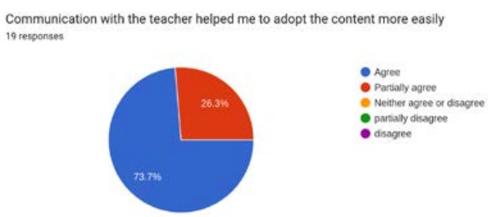

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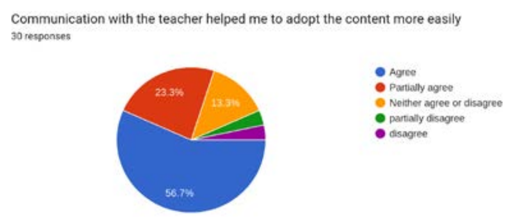
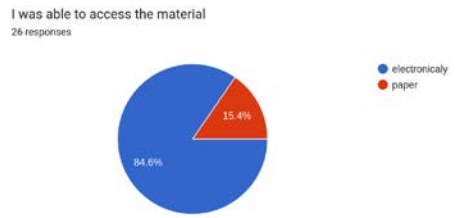


Sol_Num_Ec_Dif_ET

This another essential aspect which is relatively easily to improve. While most of the students agree or partially agree that the communication with the teacher was good, we should probably more open to the questions which the students undoubtedly have, encourage them to ask them and try to keep them involved during the teaching process.

Entry 9: I was able to access the material: electronically / paper



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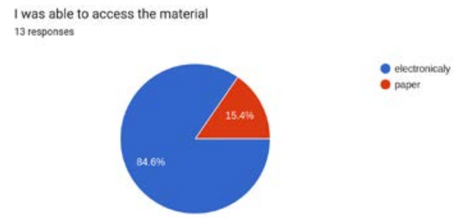

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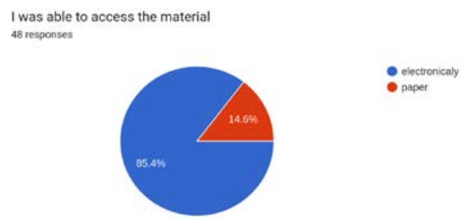

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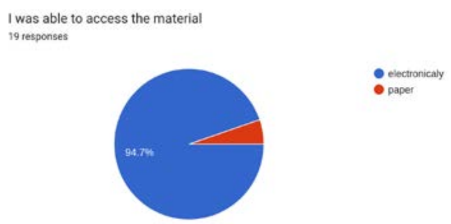
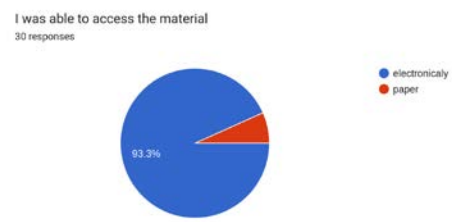


Num_Sol_Diff_Eq_ICE



Sol_Num_Ec_Dif_ET

As mentioned at Entry 7, the teaching materials available online are extensive and include everything the student needs and more. Hard copies of the courses / similar courses are available at the UPT Library. The difference between the results corresponding to the first three lessons and the ones corresponding to the last two are most probably related to the availability of the respective hard copies at the UPT Library.

6.1 Further discussion and conclusions

Overall at UPT we had **139 survey responses**:

- **26 responses** for the "Integrale improprii" Lesson from the students of the Faculty of Electronics, Telecommunications and Information Technologies
- **13 responses** for the "Integrale duble"Lesson from the students of the Faculty of Electronics, Telecommunications and Information Technologies
- **50 responses** for the "Integrale curbilinii" Lesson from the students of the Faculty of Electrical and Power Engineering
- **19 responses** for the "Numerical Solutions for Differential Equations" Lesson from the students of the Engineering in English (ICE) bachelor program at the Faculty of Civil Engineering
- **31 responses** for the "Solutii numerice pentru ecuatii diferentiale" Lesson from the students of the Faculty of Electrical and Power Engineering.

Obviously not all the attending students fulfilled the survey, but most of them did and we can consider the sample of responses a representative one.

Looking back at the experience gained by preparing and teaching these Pilot lessons, there are several conclusions which can be drawn and which can help shape future directions in the integration of STEAM methodology in everyday teaching:

- The pilot lessons at UPT were conducted both online and face-to face. The lessons included in the "Calculus 2" and "Special Mathematics" courses, regarding integral calculus, took place at the beginning of the semester, when the whole teaching activity at UPT was still online. The lessons included in the "Mathematics 4" and "Computer Assisted Mathematics" courses, regarding the numerical solutions of differential equations, took place later in the semester, when the teaching activity was already shifted to face-to-face lessons. However, we can say that in both cases the implementations of the Pilot lessons at UPT impelled us to focus more on the technology available in order to be able to communicate with the student in a more efficient and open way and in order to better illustrate and visualize the notions involved in the lectures. Here we think that an increasing role must be played by the powerful mathematical software available: Mathematica, Matlab, Geogebra etc.

- From this point of view, it seems clear, both by looking at the survey's results and by recalling private discussions with students enrolled in these courses, that the results of the increasing use of the mathematical software were highly beneficial for the students. The capability to illustrate, visualize and exemplify key notions of the lectures clearly leads to a better understanding of the topics at hand.

- While this idea of increasing the role of the mathematical software in education is a clear one and in principle relatively simple to implement, the details of the implementation depend of course on the particular topics which must be taught. In order for other teachers/researchers to be able to use the results of the project, a welcomed addition would be a library of files/programs visualizing or exemplifying the major topics included in an usual calculus course (for example, Mathematica or Geogebra files which illustrate basic concepts such as the limit of a sequence or of a function, the derivative etc.). In Mathematica the visualization may be taken to another level by the use of interactive files, which allow the user to see in real time the effect of changing one or more parameters on the final result (for example change in real time the epsilon from the definition of the limit or/and the number of terms of the sequence displayed). Here we mention the "Wolfram Demonstrations Project" (https://demonstrations.wolfram.com/), a vast repository of Mathematica files covering almost all the important topics in mathematics. Moreover, even for a casual Mathematica user is relatively easy to find a demonstration closer to the desired topic and to adapt it to the specific needs of the topic. While Mathematica is not a free software, the Mathematica Player, which allows the students to open and interact with the files proposed by the teacher, can be downloaded and used for free.

- Finally, talking about strengths and weaknesses of this teaching methodology, it is easy to see that the strengths are related to the inclusion of the current technology, which allows for an unprecedented power and flexibility of visualization. The use of online teaching platforms which permit the uploading of teaching materials and facilitate continuous contact with the students is also a positive aspect. On the other hand, if we are referring to the online teaching process, an obvious weakness is the lack of direct human contact between the teacher and the student, lack which may be the reason why the students may feel somewhat disconnected with the lessons. In order to compensate for this lack of contact, the online lessons should be, as far as the topic allows, especially engaging and attractive for the students, and this is, in our opinion, the greatest challenge for a teacher today.

7. RESULTS OF ANALYSIS AND EFFECTS OF PILOT LECTURES AT BMU

Belgrade Metropolitan University conducted pilot testing in 4 different courses, and 5 different lessons. Courses and respective lessons that were piloted were:

- Discrete structures:
  - Big O Notation
  - Recurrence Relations
- Introduction to object-oriented programming:
  - Definite Integrals in Java
- Blockchain technology in data protection:
  - Elliptic Curve Cryptography
- Artificial intelligence:
  - Classification in Machine Learning.

All courses are taken by the undergraduate students studying at the Faculty of information technology, and going for the bachelor degree in one of the tree areas: information technology, software engineering or computer games development. The goal was to introduce mathematics topics in their mathematics and non-mathematics classes, with the developed STEAM approach. Discrete structures and Introduction to object-oriented programming are thought to first and second year students, while Blockchain technology in data protection and Artificial intelligence are thought to third and fourth year students.

In total, 108 students attended the lectures, while 79 responses were collected in all courses (Table 1).

Table 1. Number of students that attended pilot lectures and the responding number of collected surveys

	No. Of students in attendance	No. of survey answers
Big O Notation	64	39
Recurrence Relations	20	17
Definite Integrals in Java	15	15
Elliptic Curve Cryptography	6	5
Classification in Machine Learning	3	3

7.1 Preparation of pilot lessons

Pilot lessons were prepared taking into consideration both developed recommendation for STEAM approach within the project, as well as the institutional requirements that also have to fulfill at the University. Lesson contents were structured around the following principles:

- Lesson plan design was first created
- Lesson content was designed following the lesson plan
- Beginning of the lesson contains:
  - Clear statement of lesson objectives and learning outcomes, which highlights the expectations that students should have from learning the lesson
  - States if the lesson requires any previous knowledge
  - Motivation for students formulated in the form of “interest catcher” giving the description of the practical problem (“real world” problem/application)
- Theoretical/instructional content,
- Lesson videos,
- Activating exercises, such as discussion, self-evaluation with feedback, etc.
- Assessments – each lesson contains multiple self-assessment tests (minimum of 2)
- Student homework assignment or problem assignment.

At BMU, given that students are studying engineering programs, and specifically in computing disciplines, it is hard for students to understand why they have to study mathematics, other than the typical answer “it is good for developing logical thinking of engineers.” Calculus topics are foundation for any engineers, and so it is for BMU students, but for students it is hard to see how calculus topics directly relate with that they will have to learn in the future. This is why, it was important for BMU and realization of this project to be able to relate “real world problems” from software engineering, information technology and video games, with calculus topics. This is the reason why certain topics were chosen, to be able to cover all these computing disciplines. Examples of interest catchers are given in Table 2.

Table 2. Examples of interest catchers for pilot lessons.

Lesson title	Interest catcher with “real world” problems/applications
Recurrence relations	Examples of recursion in programming, A mathematical example of recursion is the definition of factorials, The tower of Hanoi
Big O	How does one go about analyzing programs to compare how the program behaves as it scales?
Classification in machine learning	Classification of Iris dataset using Bayesian classifier.
Definite integrals solving in Java	Software implementation of integration methods
Elliptic curve cryptography	How can blockchain use less processing (and therefore electrical) power?

Developed lessons are published on Learning Management Systems so that students can access content, both interactively, or download it in PDF format. Interactive lesson contains mind maps for students easier navigation through the learning content, as well we the lesson videos, and self-assessments. On the other hand, PDF format of the lesson contains all of the learning content, without the active learning materials (videos, assessments, forum discussions, etc.). Making the lesson available online to students, it was possible to follow student progress and activity.

Examples of conducted pilot lessons

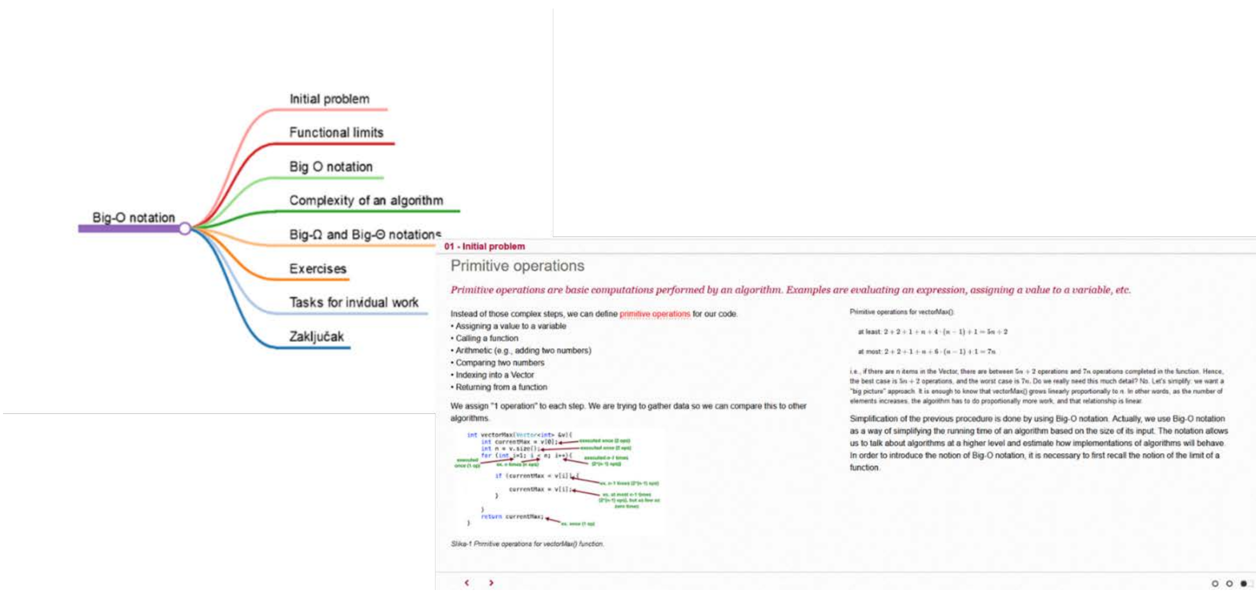
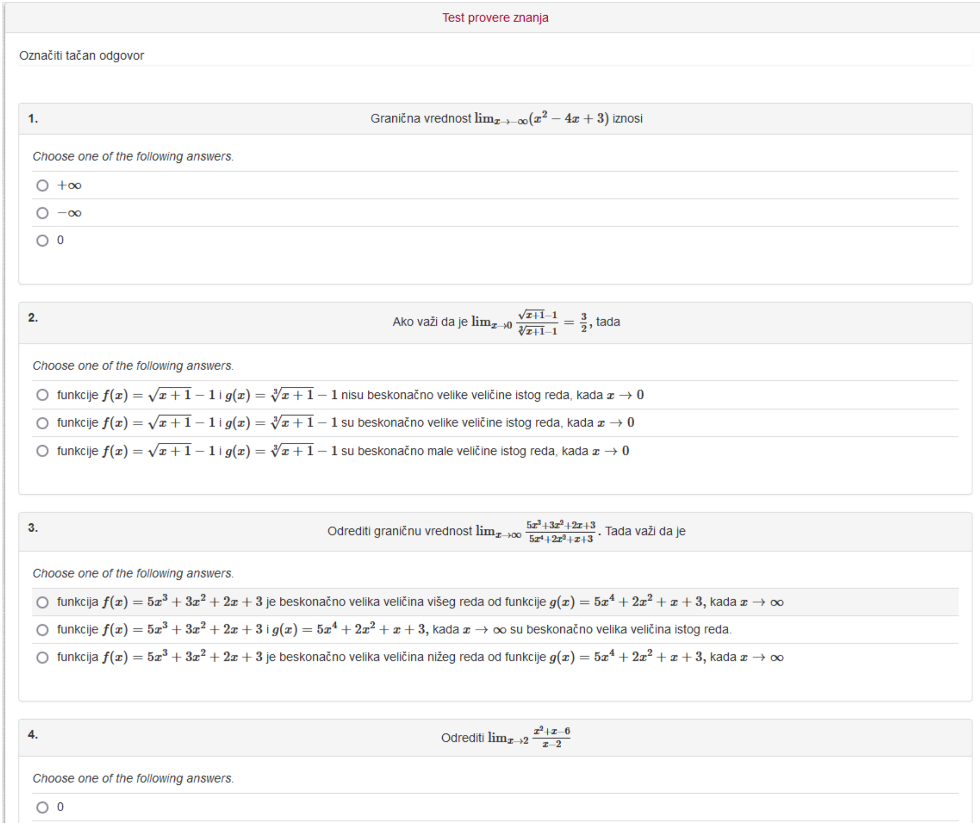


Figure 36. Example of a mind map and one part of the lesson content published on LMS for lesson “Big O notation”


Test provere znanja

Označiti tačan odgovor

1. Granična vrednost $\lim_{x\to-\infty}(x^2-4x+3)$ iznosi

Choose one of the following answers.

- $+\infty$
- $-\infty$
- 0

2. Ako važi da je $\lim_{x\to 0}\frac{\sqrt{x+1}-1}{\sqrt[3]{x+1}-1}=\frac{3}{2}$, tada

Choose one of the following answers.

- funkcije $f(x)=\sqrt{x+1}-1$ i $g(x)=\sqrt[3]{x+1}-1$ nisu beskonačno velike veličine istog reda, kada $x\to 0$
- funkcije $f(x)=\sqrt{x+1}-1$ i $g(x)=\sqrt[3]{x+1}-1$ su beskonačno velike veličine istog reda, kada $x\to 0$
- funkcije $f(x)=\sqrt{x+1}-1$ i $g(x)=\sqrt[3]{x+1}-1$ su beskonačno male veličine istog reda, kada $x\to 0$

3. Odrediti graničnu vrednost $\lim_{x\to\infty}\frac{5x^3+3x^2+2x+3}{5x^4+2x^2+x+3}$. Tada važi da je

Choose one of the following answers.

- funkcija $f(x)=5x^3+3x^2+2x+3$ je beskonačno velika veličina višeg reda od funkcije $g(x)=5x^4+2x^2+x+3$, kada $x\to\infty$
- funkcije $f(x)=5x^3+3x^2+2x+3$ i $g(x)=5x^4+2x^2+x+3$, kada $x\to\infty$ su beskonačno velika veličina istog reda.
- funkcija $f(x)=5x^3+3x^2+2x+3$ je beskonačno velika veličina nižeg reda od funkcije $g(x)=5x^4+2x^2+x+3$, kada $x\to\infty$

4. Odrediti $\lim_{x\to 2}\frac{x^2+x-6}{x-2}$

Choose one of the following answers.

- 0

Figure 37. Example of self-assessment test in the lesson “Big O notation”

Figure 38. Example of part of class for the lesson “Big Bo notation”

Figure 39. Example of part of class for the lesson “Elliptic curve cryptography”

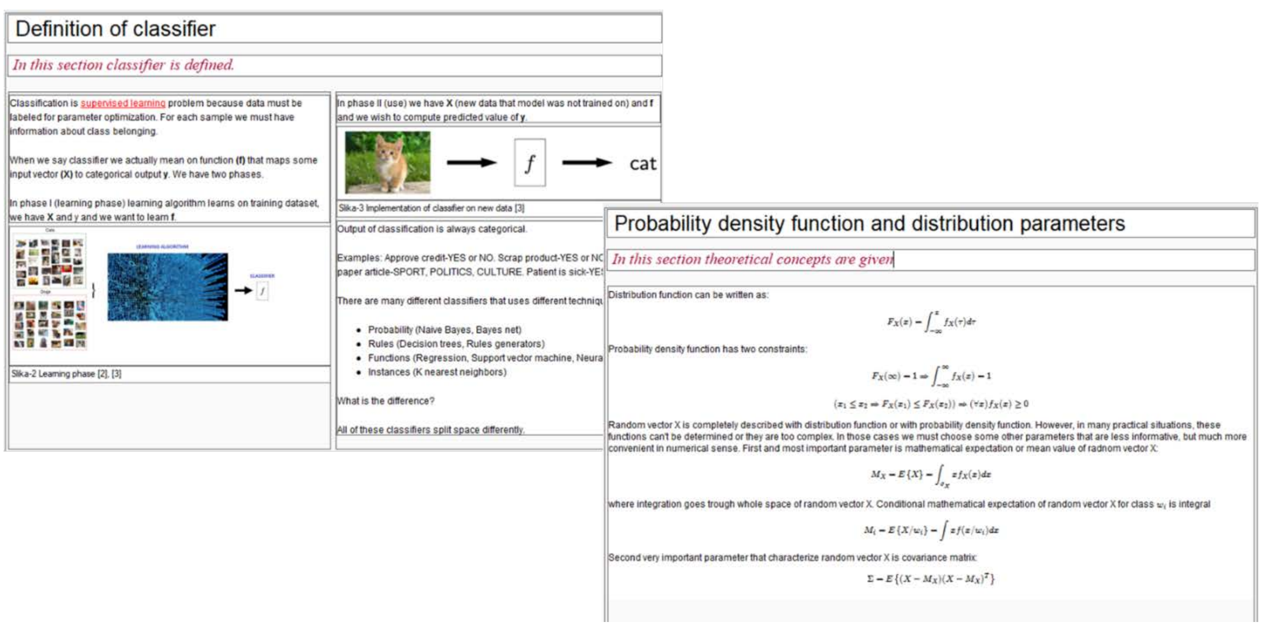


Figure 40. Example of lesson content for “Classification in machine learning”

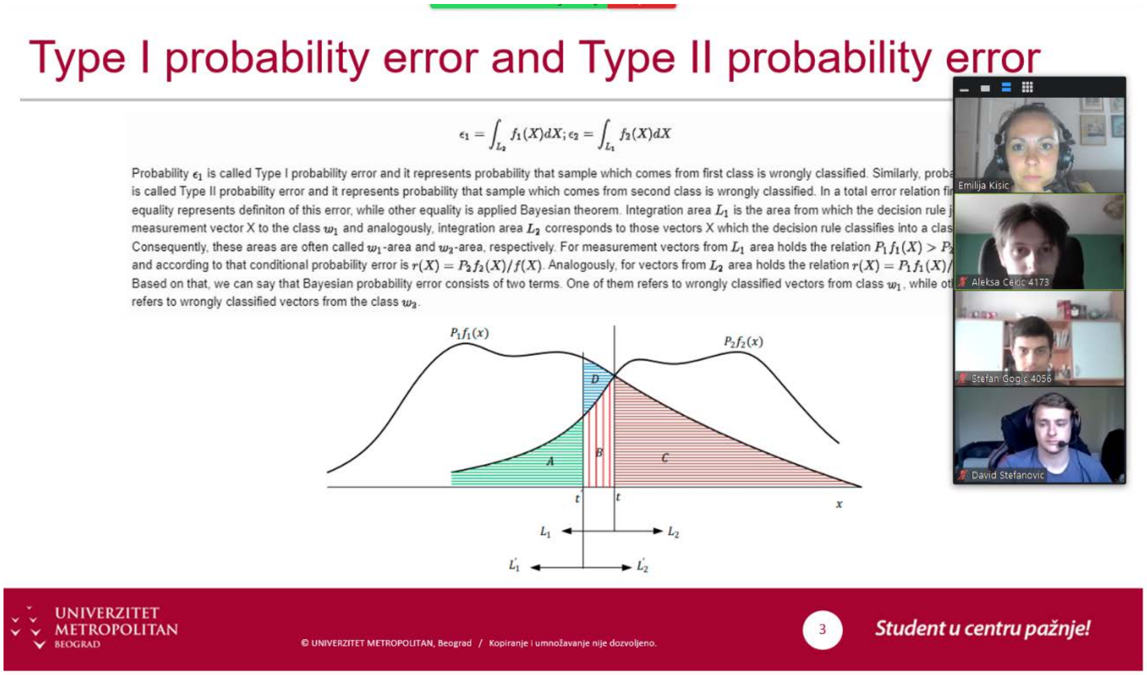


Figure 41. Example of part of the class for “Classification in machine learning”

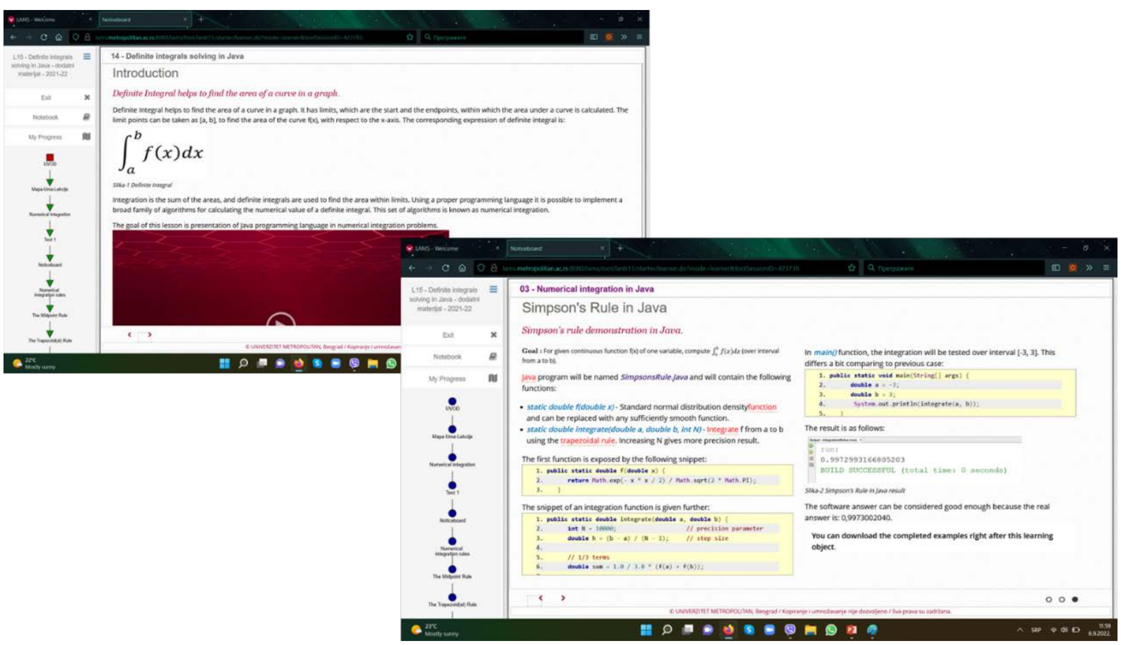


Figure 42. Examples of lesson content for the lesson “Definite integrals in Java”

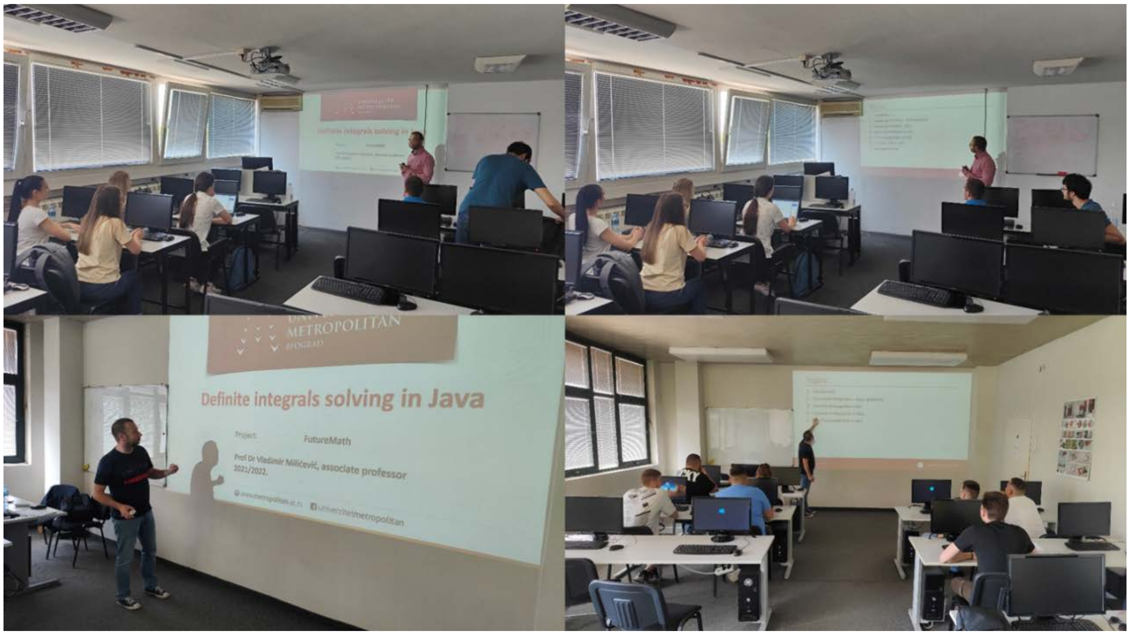


Figure 43. Example of parts of classes for the lesson “Definite integrals in Java”

7.2 Overview of the Survey

Summary of student answers for all 5 pilot lessons are shown in Table 3. Table 4 shows a more statistical analysis of the answers given by the students, showing the mean, standard deviation and median.

Table 3. Student survey answers for all 5 pilot lessons

	I agree	I partially agree	I neither agree nor disagree	I partially disagree	I disagree
Q1: The lesson was well organized	87%	11%	1%	0%	0%
Q2: Computer environment helped me to get visual approach of mathematics contents	70%	19%	10%	1%	0%
Q3: Visualization helped me to acquire knowledge more easily	76%	23%	1%	0%	0%
Q4: Teaching contents are interesting	62%	25%	8%	5%	0%
Q5: Teaching contents are applicable in everyday life	59%	24%	14%	1%	1%

Q6: Teaching contents are applicable in sciences	61%	23%	9%	5%	1%
Q7: Literature is adequate for understanding the teaching contents	68%	24%	5%	3%	0%
Q8: The communication with the teacher helped me to acquire knowledge more easily	85%	9%	3%	0%	4%

Table 4. Results of survey questions

Questions	**Mean ±**	**Standard deviation**	**Median**
Q1: The lesson was well organized	4.86	0.38	5.00
Q2: Computer environment helped me to get visual approach of mathematics contents	4.57	0.73	5.00
Q3: Visualization helped me to acquire knowledge more easily	4.75	0.47	5.00
Q4: Teaching contents are interesting	4.44	0.84	5.00
Q5: Teaching contents are applicable in everyday life	4.39	0.87	5.00
Q6: Teaching contents are applicable in sciences	4.39	0.94	5.00
Q7: Literature is adequate for understanding the teaching contents	4.58	0.71	5.00
Q8: The communication with the teacher helped me to acquire knowledge more easily	4.71	0.85	5.00
Q9: Teaching material is available in the form: printed or electronic	0.97	0.16	1.00

The discussion about each question survey follows:

Q1. The lesson was well organized

Most of the students thought that lessons were well organized. 87.34% of students agreed, 11.39% of students partially agreed, while only 1 student (1.27%) neither agreed nor disagreed.

Q2. Computer environment helped me to get visual approach of mathematics contents

In all courses computer environment was used in different ways. While all lessons were presented to students using university's Learning Management System, the assigned tasks and problems required using different software tools. The least additional software tools (other than LMS) were used in the course Discrete structures, and this is where all of the “I neither agree nor disagree” answers come from. In total 69.62% of students agreed that computer environment helped them in visualizing mathematical content, while 18.99% partially agreed, 10.13% neither agreed nor disagreed, and only 1 student (1.27%) partially disagreed.

Q3. Visualization helped me to acquire knowledge more easily

All lessons were taught live at the University’s campuses, and each presenter had a visual aid, such as a projector and screen and speakers. All lessons were accompanied with a presentation, assigned tasks and problems, assigned tests and other learning materials relevant to the lesson. In total, 75.95% of students agreed, 22.78% partially agreed, and 1.27% neither agreed nor disagreed.

Q4. Teaching contents are interesting

All planned lessons were presented with a reflection on contemporary real-world problems, mainly relating to computing disciplines. Some of the students did not actively participate in the reflection; however, most of the students were excited to join in the conversation and work on assigned tasks.

In total, 62.03% of students agreed, 25.32% partially agreed, 7.59% neither agreed nor disagreed, and 5.06% partially disagreed.

Q5. Teaching contents are applicable in everyday life

While some of the topics were too abstract at first for some of the students, the presenters gave multiple real-world scenarios in which the solutions deriving from the topics presented could be applied. After presenting real-world applications, students have more eager to follow the lesson.

In total, 59.49% of students agreed, 24.05% partially agreed, 13.92% neither agreed nor disagreed, 1.27% partially disagreed, while 1.27% disagreed.

Q6. Teaching contents are applicable in sciences

Following from Q5, most students argued that some of the topics could be better suited primarily for scientific application, rather than for real-world problems. This is the reason the previous question and this one have similar answers.

In total, 60.76% of students agreed, 22.78% partially agreed, 8.86% neither agreed nor disagreed, 5.06% partially disagreed, while 2.53% disagreed.

Q7. Literature is adequate for understanding the teaching contents

All lessons were presented using the University’s Learning Management System, which is suited for undergraduate-level studies. In addition to the full lesson content, students were given additional useful literature that can be used to further expand their knowledge.

In total, 68.35% of students agreed, 24.05% partially agreed, 5.06% neither agreed nor disagreed, 2.53% partially disagreed, while 0% disagreed.

Q8. The communication with the teacher helped me to acquire knowledge more easily

All teachers have multiple years of experience in teaching as well as in scientific research in the field related to the topics of the lesson. In total, 84.81% of students agreed, 8.86% partially agreed, 2.53% neither agreed nor disagreed.

Q9. Teaching material is available in the form: printed or electronic

Teaching material was made available to students online via university's LMS system. No print outs were distributed to students, so it is not clear why 2 answers out of 79, pointed towards teaching materials being available in the printed format. One of the possibility is that students printed out teaching materials themselves.

In addition, it should be noted that following topics are important to discuss as far the pilot lessons:

10. Collaborative learning was applied successfully

This question was not in the survey; however, students engaged in conversation after each segment of the given topic was presented. After a short break between classes, students suggested several scenarios in which some of the topics could be applied.

11. Project based learning helped me to acquire knowledge more easily

This question was not in the survey, but at the University, each of the students has to prepare and defend a project for each class at the end of the semester; it was no different for the given classes.

7.2 Discussion about the experience of integrating STEAM principles at BMU

1. How many answers did you get to the survey?

In total, 108 students attended the lectures, while 79 responses were collected in all courses.

2. What was achieved by implementing the pilot lectures for students, teachers and your organization?

At BMU, given that students are studying engineering programs, and specifically in computing disciplines, it is hard for students to understand why they have to study mathematics, other than the typical answer "it is good for developing logical thinking of engineers." Calculus topics are foundation for any engineers, and so it is for BMU students, but for students it is hard to see how calculus topics directly relate with that they will have to learn in the future. This is why, it was important for BMU and realization of this project to be able to relate "real world problems" from software engineering, information technology and video games, with calculus topics. This is the reason why certain topics were chosen, to be able to cover all these computing disciplines.

3. How much practical knowledge does students gained from the new teaching methodology?

Students in the first and second year were given fundamentals they will need in the later years. Even though, all lesson are motivated with real world problems, several students from this group are still having hard time relating calculus topics to real world problems. On the other hand, third and fourth year students are seeing easier this connection, and the courses they were taking where the pilot lessons were conducted are already focused on applied knowledge.

4. Are there other important findings emerging from the implementation of the new teaching methodology developed within the project that you would like to mention.

Our findings showed that, in order to implement the new teaching methodologies, it is favorable to start with first year students, as students in the following years of study quickly lose interest if not acquainted with the principles. Furthermore, good LMS integration is necessary for the STEAM principles to be included in the teaching materials as better as possible.

5. Is it possible for teachers/researchers outside the project to use the results of the project and the newly developed teaching methodology?

The methodology of the FUTUREMATH project is nicely set up, so that it can be used not only in the calculus or mathematics courses, but rather can be expended to many different topics. Relating lesson to topics that students find interesting is an interest catcher for them. Also, videos, tests, assignments, and other materials that are included in the lesson, provide students with variety of information, and keeps them interested in the topic.

6. What are the major strengths of this teaching methodology?

   The major strengths of this teaching methodology can be summarized as follows:

   1. Real-world problems can be easily implemented as examples during classes, making the topics easier for students to understand,
   2. The inclusion of various additional materials in different formats allows students to be interested in the given topics,
   3. More abstract and complex mathematical problems can be presented in a manner that keeps students active during and after classes

7. What are the major weaknesses of teaching methodology?

   The major weaknesses of this teaching methodology can be summarized as follows:

1. Part of students still have guard towards mathematics topics, and have a hard time relating necessity of acquiring mathematics foundation for future profession of engineers.
2. When teaching to first year students, when fundamentals are necessary to be covered for more advanced topics, it is challenging to relate mathematics to the full extent, other than mentioning how these principles will be later applied in their courses and field of study.

8. Based on the experiences of this project make recommendation on how integrate STEAM principles into STEM studies.

Based on the experiences at BMU, we can conclude that integrating STEAM principles into STEM studies can be very beneficial for keeping the interest and motivation among students, to better understand mathematics fundamentals that they need and will need in their profession. This approach does acquire additional effort from teachers and a level of creativity when introducing active learning methodologies and STEAM principles. When appropriate, students should be taught with STEAM principles from their beginning of studies, to get them acquainted with the principles as well, so that later they would be accustomed to this type of learning.

8. GENERAL FRAMEWORK FOR INTEGRATING STEAM PRINCIPLES FOR STEM STUDIES

In this chapter first we will make a general analysis on the effects of the pilot lectures of the four partner universities. Based on the finding of this analysis, a general framework for integrating STEAM principles for STEM studies will be given.

8.1 General analysis on the effects of the pilot lectures of the four partner Universities

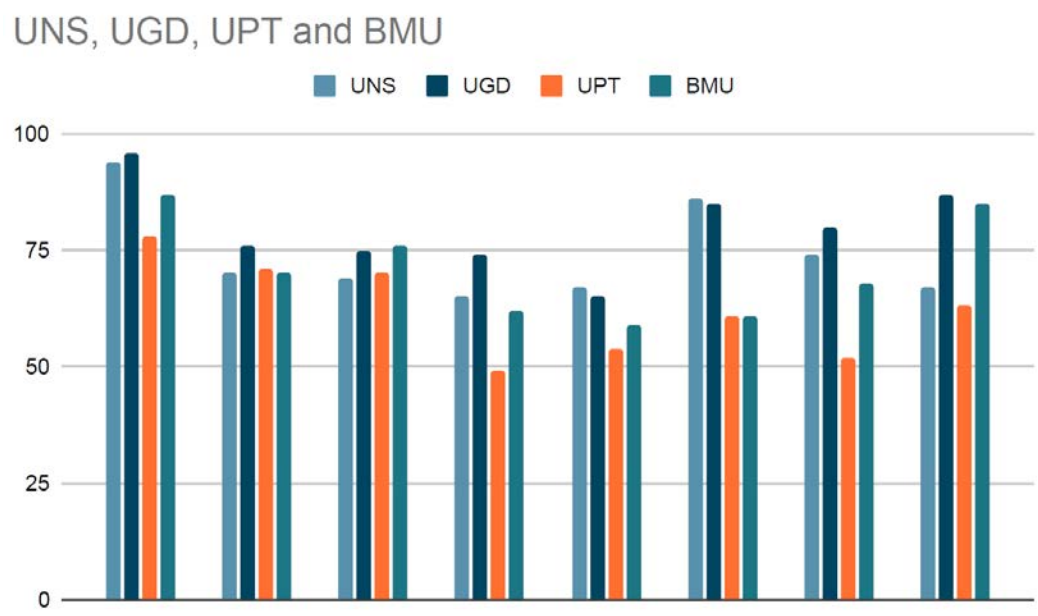
We got 401 answers for all piloting lessons conducted at the partner Universities included in this project. We consider this number of students to be relevant for further analysis and general conclusions.

The general analysis of the four universities surveys can be seen in the following table.

Table 1. Percentage of students that agree with the question.

		UNS	UGD	UPT	BMU	
1	The course was well organized	94	96	78	87	88.75
2	Computer environment helped me to get visual approach of mathematics contents	70	76	71	70	71.75
3	Visualization helped me to acquire knowledge more easily	69	75	70	76	72.5
4	Teaching contents are interesting	65	74	49	62	62.5
5	Teaching contents are applicable in everyday life	67	65	54	59	61.25
6	Teaching contents are applicable in sciences	86	85	61	61	73.25
7	Literature is adequate for understanding the teaching contents	74	80	52	68	68.5
8	The communication with the teacher helped me to acquire knowledge more easily	67	87	63	85	75.5

Figure 44. Graphical representation of the students that agree on the questions



By implementing the FutureMath project and the pilot lectures at the partner universities the students obtained a different approach to problem solving, compared to the classical way of teaching. The use of the technology in the teaching was very much included. Students were given a practical real-life problem to think on, and then to solve it. The methods for solving the problem are made step by step in order to be easily understood by the students. During this process, students have built problem-solving skills and teamwork, as well as leadership skills, all of which are necessary for life outside of classroom. Even 88,75% of students from the partner universities, who took part in fulfilled the questionnaire are believe that the courses were well organized.

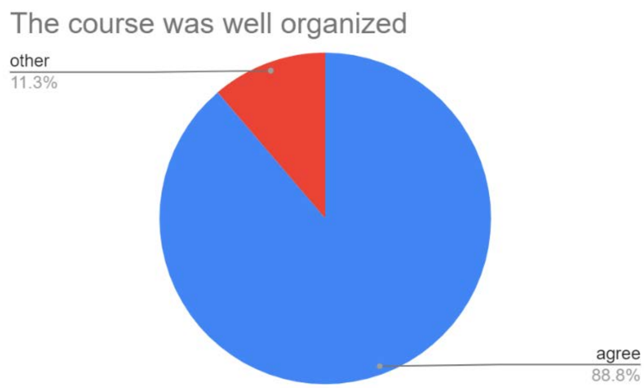


Figure 45. The course was well organized

The new teaching methodology actively involves use of technology. Computer programs like Mathematica, MathLab, Geogebra were used for better visualization of mathematics context.

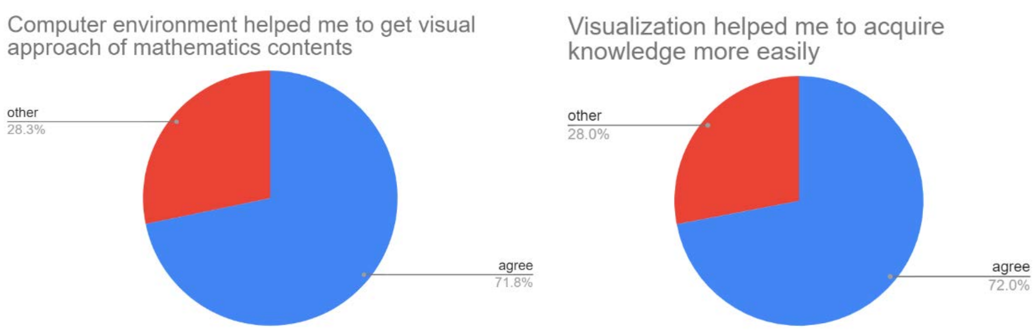


Figure 46. Computer environment helped me to get visual approach of mathematics contents. Visualization helped me to acquire knowledge more easily

The teachers, taking part in piloting were well prepared for the courses. They prepared lectures using the already developed STEAM methodology. All the lectures are public, published on the Web pages of the Universities, so they can be used by other stakeholders and other parties.

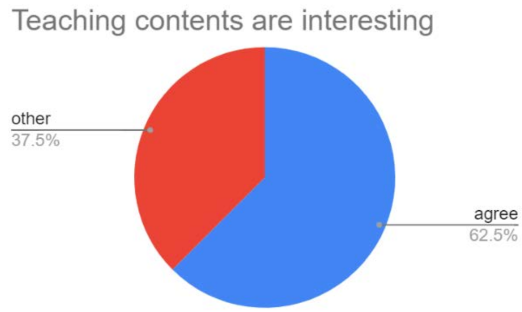


Figure 47. Teaching context are interesting.

The new teaching methodology was all about the practical knowledge of the students. A lot of practical applications in the frame of STEAM were given to the students. Collaborative learning, together with project and problem based learning and mathematical modeling contribute to students' practical knowledge. Around 62 % of the students answered that the new approach and the new teaching methodology had a strong connection with application to the real-life problems, and application to sciences.

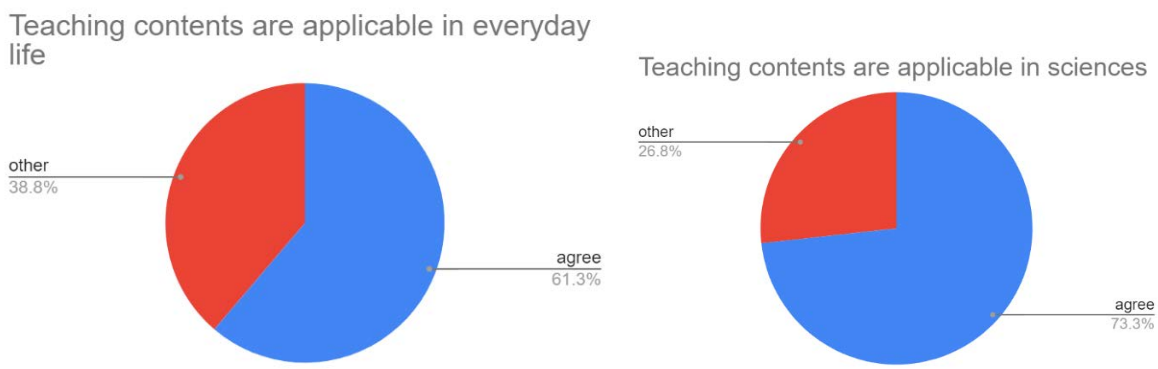
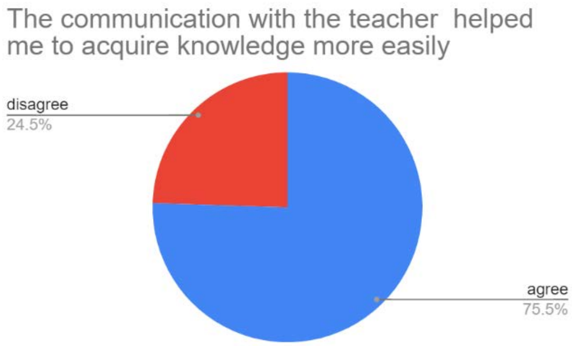


Figure 48. Teaching contents are applicable in everyday life and sciences

All the students stated that the lecture was well prepared, meaning that it was interesting for them to follow. 75.5% of the students think that the communication with the teacher helped them to grasp the contents more easily, and 68.5 % agree that the literature for the study is well adapted.



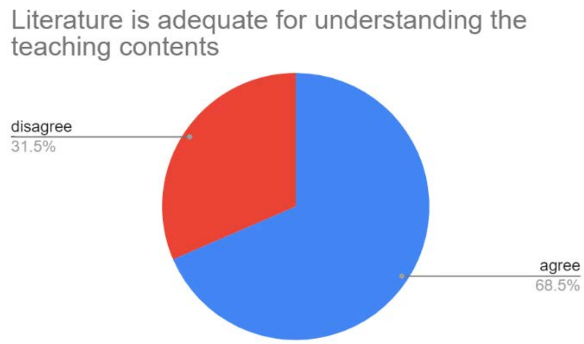


Figure 49. The communication with the teacher helped me to acquire knowledge more easily
Literature is adequate for understanding the teaching contents

The results of FutureMath project, developed proposals, guidelines, calculus courses, STEAM intervened calculus topics, are prepared in English, Serbian, Macedonian and Romanian language and placed of project Webpage. Developed teaching STEAM methodology collaborative learning, project and problem-based learning as well as mathematical modeling are presented on Webpage. All of them can be used for the teachers and researches from other universities and other students and teachers interested in this teaching material.

8.2 A framework for integrating STEAM principles for STEM studies

European nations invest in innovation to promote sustainable economic growth. While many countries are suffering from the effects of global economic difficulties, such as rising unemployment and soaring public debt, the role of labor input is decreasing in the 21st century economy. Only innovation-driven growth has the potential to create value-added jobs and industries. Because innovation is largely derived from advances in the science, technology, engineering, and mathematics (STEM) disciplines, an increasing number of jobs at all levels require STEM knowledge.

Innovation involves the integration of diverse STEM skills and transcends disciplines. Innovation is a highly interactive and multidisciplinary process that rarely occurs in isolation and is tightly connected to life and real-life problems.

Innovation is needed in the curricula of undergraduate studies as well. Though this project we developed and tested a new teaching methodology that in a innovative way give the students interesting approach, more visualization of the problems to be solved, and involve computers and new technology.

Traditional learning sees a teacher at the front imparting knowledge to a whole class of learners who are listening and taking notes. In contrast, STEM approaches require students to actively participate in their own knowledge acquisition. In turn, the teacher becomes a guide or mentor rather than a subject expert. Instead of one focal point in a classroom, there may be many.

At the end, we can summarize and define the following framework for integrating STEAM principles for STEM studies.

- Cross-curricular and multi-disciplinary approach.
- Involves critical thinking: How to argue a case, evaluate facts and reach conclusions.
- Collaboration: Working in teams, forming team roles, building consensus
- Real-life problem solving - Practical not theoretical
- Involves visualization: to understand by visual means
- Involves technology: since the technology is present and it cannot be ignored.

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Appendix 1. Additional pilot lessons

<table>
<tr><th colspan="3">TOPIC PLAN</th></tr>
<tr><td>Partner organization</td><td colspan="2">UNS</td></tr>
<tr><td>Topic</td><td colspan="2">Functions of Several Variables</td></tr>
<tr><td>Lesson title</td><td colspan="2">Applications of directional derivatives</td></tr>
<tr><td>Learning objectives</td><td>✓ Students will be able to determine directional derivatives of functions of several variables, gradient;
✓ Students will acquire and deal with derivatives of a function;
✓ Students will be able to deal with different problems in everyday life, which require finding directional derivatives of a given function;
✓ Students will learn to use their mobile phones as a helping tool in solving mathematical problems</td><td rowspan="2">Strategies/Activities
☐Graphic Organizer
☐ Think/Pair/Share
☐Modeling
☐Collaborative learning
☐Discussion questions
☐Project based learning
☐Problem based learning

Assessment for learning
☐Observations</td></tr>
<tr><td>Aim of the lecture / Description of the practical problem</td><td>Introducing students to several applications of directional derivatives and the gradient
Pratical problem is to find different slopes on the surface, maximal and minimal slopes and isohypses</td></tr>
</table>

<table>
<tr>
<td>Previous knowledge assumed:</td>
<td>Basic vector calculus
Differential calculus
Definition of partial derivatives
Calculation of partial derivatives</td>
<td>☐Conversations
☐Work sample
☐Conference
☐Check list
☐Diagnostics</td>
</tr>
<tr>
<td>Introduction / Theoretical basics</td>
<td>

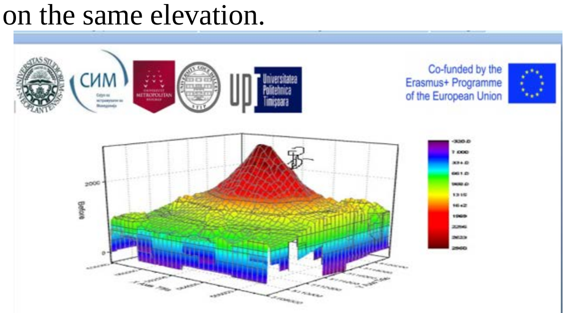
In the introduction the notion of directional derivatives Is recalled. After that a couple of basic examples are calculated.

1. Find the gradient directional derivatives of the function
$$z| = x^3 - xy^2$$
From In the point A(1,1) find the values of partial derivatives and directional derivatives in the direction of

a) Vector $\vec{a} = (3,4)$
b) $towards\ the\ point\ B(-1,2)$.
c) In the direction of Points C(1,2(, D(2,1)
The students should notice that the solution of c) is the same as the partial derivative values and acknowleging that partial derivaties can be seen as a special directional derivetives.

The the teacher gives the next problem to the students:

1. Suppose that the height of a hill above sea level is given by $z = 4 - 0.1x^2 - 0.2y^2$. If you are at the point (60,100) in what direction is the elevation changing fastest? What is the maximum rate of change of the elevation at this point?
Afetr solving this problem students are directed towards a 2D and 3D heat map and the notion of isohypse is recalled. Using the surface from the previous problem the natural question of which direction should the mountaineer move in order to stay on the same elevation.



A rough sketch model of a mountainneer

After solving this problem students are divided into groups. They are given a task to visualize the previous problem their mobile phones try to visualize the surface from They are

</td>
<td>

Assessment as learning
☐Self-assessment
☐Peer-assessment
☐Presentation
☐Graphic Organizer
☐Homework

Assessment of learning
☐Test
☐Quiz
☐Presentation
☐Project
☐Published work

</td>
</tr>
</table>

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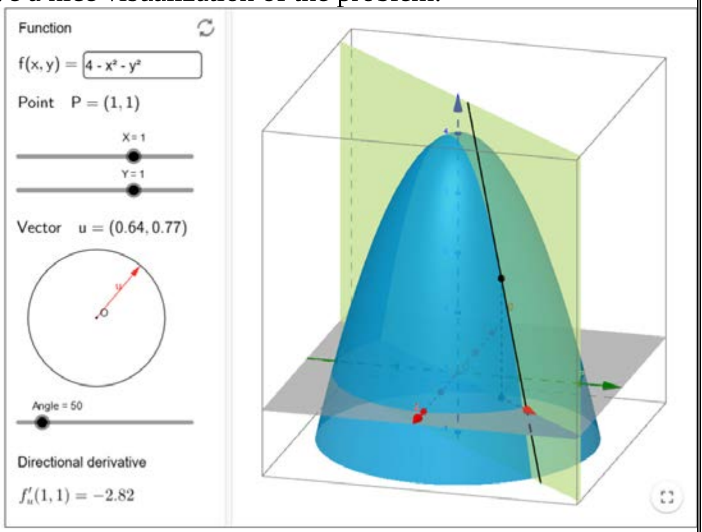
	directed to several websites and Geogebra animations that give a nice visualization of the problem.  At the end of the class students compare their solutionsand discuss them among themselves.	
Action	Using differentiating techniques and basic vector calculus after viusalising a problem.	
Materials / equipment / digital tools / software	*The materials for learning* are given as a part of references of the end from this topic plan; *Equipment*: classroom, whiteboard, marker in different colours; *Digital tools*: laptop, projector; *Software*: Geogebra	
Consolidation	With the given examples students can consider that the real functions and their derivatives are important for solving real life problems. Students will learn what is a directional derivative of a function and gradient and how to calculate it. They can learn how to apply directional derivatives in a real problem. Students can use technology, different digital tools and software as a help for solving problems, but can also realize that even with technology, solving different everyday problems is difficult without math knowledge.	
Reflections and next steps		
Activities that worked		**Parts to be revisited**

Problem solving, collaboration, using technology	Depends on the students, in a conversation with students the teacher will realize the difficulties that students had and then revisit appropriate parts.

References

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[4] www.geogebra.org

Topics plan		
Partner organization	University of Novi Sad	
Course	Programming 1	
Lesson title	Combinatorics: Splitting the numbers into sum, variations with repetitions, permutations	
Date, time	May 4th, 11:30-14 UNS,	
Learning objectives	• Students will understand the methods of splitting the number into the sum of the k numbers, for given $k \in \mathbb{N}$. • Students will understand how to generate all the variations of the given set with repetitions. • Students will understand how to generate all the permutations of the given set. • Students will understand how to implement the methods in Python programming language. • Students will understand how to apply the algorithms in solving similar combinatorial problems.	**Methodology** ☐Modeling **X Collaborative learning** ☐Project based learning **X Problem based learning** **Strategies/Activities** ☐Graphic Organizer ☐Think/Pair/Share ☐Discussion questions **Assessment for learning** **X Observations** **X Conversations** ☐Work sample ☐Conference ☐Check list ☐Diagnostics **Assessment as learning** **X Self-assessment** ☐Peer-assessment ☐Presentation
Aim of the lecture / Description of the practical problem	The aim of the lecture is to make students able to use Python in solving combinatorial problem, with visual solutions. As a practical problem, the lecturer poses several questions related to the applications of combinatorial methods in real life situations.	
Previous knowledge assumed:	• Elementary programming skills in Python. • Basics of Combinatorics.	
Lecture	In the introduction, we give basic examples of the usage of recursion, and the connection between recursive formula in mathematics, with application to natural problems in biology and how we solve it through computers (STEAM).	

As the starting example, we use Fibonacci sequence, and relate it to the golden ratio, and give an example where it can be found.

We continue with recursive combinatorial problems that can be easily implemented in Python:

1. splitting the given number as the sum of non-negative integers, where we use the recursive description of the structure $x_0 + x_1 + \cdots + x_{k-1} = s$, where s and k are given, that for x_{k-1} we can use any of the numbers $0, 1, 2, \ldots, s$, and the rest forms the splitting of the number $s - x_{k-1}$ into $k - 1$ numbers.

We write the following code in Python, after which we give some further examples, like not knowing the number of the summands in advance, but they have to be positive integers, and some others.

```
import numpy as np

def ispisi_sabirke(sabirci):
    print(sabirci[0], end="")
    for i in range(1, len(sabirci)):
        print(" + {0}".format(sabirci[i]), end="")
    print()


def razbij_u_zbir(sabirci, s, k):
    if k == 1:
```

☐Graphic Organizer
☐Homework

Assessment of learning
X Test
☐ Quiz
☐Presentation
☐Project
☐Published work

```
        sabirci[0] = s
        ispisi_sabirke(sabirci)
    else:
        for i in range(s+1):
            sabirci[k - 1] = i
            razbij_u_zbir(sabirci, s - i, k - 1)

k = 3
s = 4
sabirci = np.empty(k, int)
razbij_u_zbir(sabirci, s, k)
```

2. Next, we consider the variations with repetition of length k, of the given set of n elements, S_n, which is the ordered k-tuple of the elements of that set. For the set we take $\{0, 1, \dots, n-1\}$. To be able to generate all the variations with repetitions, we use the recursive description of the structure: the last element can be any element of the given set, and before this element we can put any variation with repetition of the set $\{0, 1, \dots, n-1\}$ of length $k-1$.

In Python, the following code executes the aforementioned.

```
import numpy as np


def vsp(niz, n, k):
    if k == 0:
        print(niz)
    else:
        for i in range(n):
            niz[k - 1] = i
            vsp(niz, n, k - 1)


n = 2
k = 4
niz = np.empty(k, int)
vsp(niz, n, k)
```

3. Lastly, we consider all the permutations of the given set with n elements, S_n, is any n −tuple of different elements from that set. We start with the set $\{0, 1, \dots, n-1\}$. To be able to go through all the permutations, we use the recursive description of this structure: the first element can be any element of the given set, after which we can place any permutation of the remaining elements. In Python, this looks as follows.

```python
import numpy as np


def zameni(niz, a, b):
    niz[a], niz[b] = niz[b], niz[a]


def per(niz, n, m):
    if m == n:
        print(niz)
    else:
        for i in range(m, n):
            zameni(niz, m, i)
            per(niz, n, m + 1)
            zameni(niz, m, i)


n = 4
niz = np.arange(n)
per(niz, n, 0)
```

Action	The demonstration of power of Python in solving the combinatorial problems and visualization.	
Materials / equipment / digital tools / software	Computer, electronic whiteboard, PyCharm software	

Reflections and next steps

Reflections	**Next steps**
The attendance was average due to the fact that we were working under certain epidemiological restrictions; the feedback was positive; the results of the tests show that the students have benefited from the materials and equipmebt used to deliver the lecture	Since this approach was successfully implemented and was well received, the next steps include implementing other parts of the curriculum using the strategies devised for the pilot lecture.

References

In Apendix:
Photographs, Lists of students, Test, questionare

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<table>
<tr><th colspan="3">TOPIC PLAN</th></tr>
<tr><td>Partner organization</td><td colspan="2">Goce Delcev University – Stip, North Macedonia</td></tr>
<tr><td>Topic</td><td colspan="2">Double Integrals: Calculating Area</td></tr>
<tr><td>Lesson title</td><td colspan="2">Calculating Area Using Polar Coordinates</td></tr>
<tr><td>Learning objectives</td><td>✓ Students will acquire and deal with double integrals;
✓ Students will be able to estimate area of different 2D shapes, including shapes which border is constructed with circles;
✓ Students will be able to deal with different problems in everyday life, which require calculating area;
✓ Students are encouraged to use technology and different software in their work, while considering problem - based situations.</td><td rowspan="2">Strategies/Activities
☐Graphic Organizer
☐Think/Pair/Share
☐Modeling
☐Collaborative learning
☐Discussion questions
☐Project based learning
☐Problem based learning

Assessment for learning
☐Observations
☐Conversations
☐Work sample
☐Conference
☐Check list
☐Diagnostics

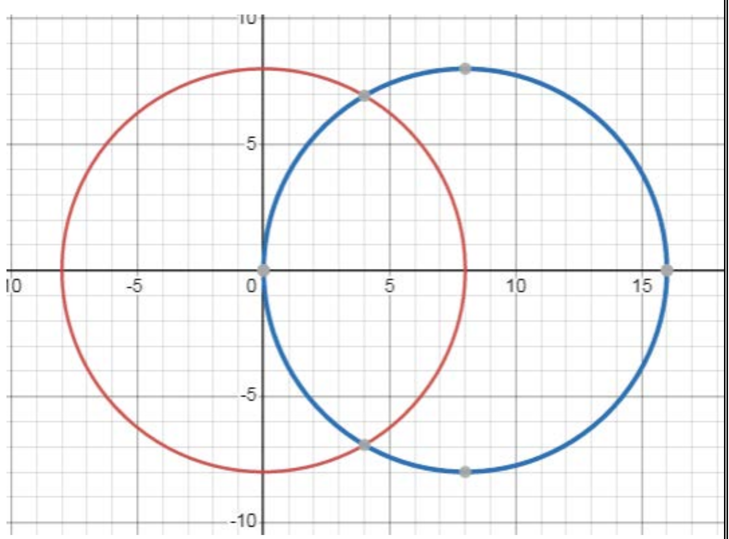
Assessment as learning
☐Self-assessment
☐Peer-assessment
☐Presentation
☐Graphic Organizer</td></tr>
<tr><td>Aim of the lecture / Description of the practical problem</td><td>The aim of the lecture is to make students able to calculate area of 2D - shapes which border is constructed with circles. It is easier to be done using polar coordinates.

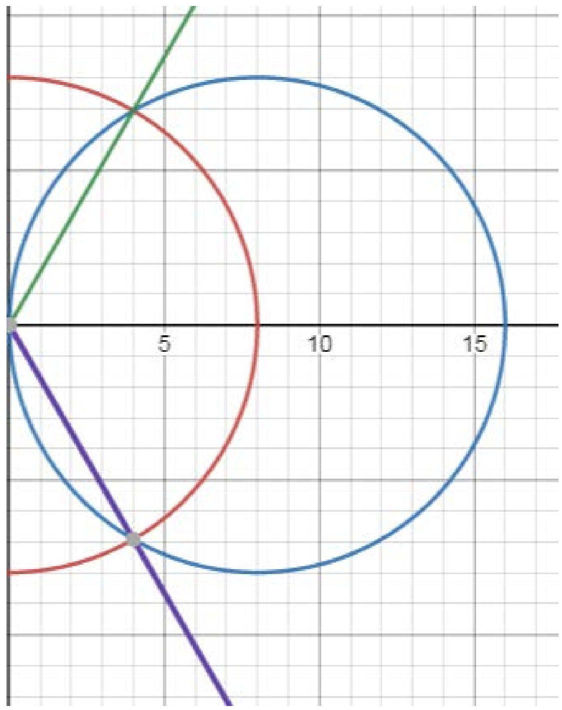
The teacher gives the next problem to the students:

A children's playground has the shape of two circles with equal radius R=8 meters and central distance d=8 meters. In the intersection of the two circles, there are children's requisites for playing, and outside the intersection, on both sides, there is a green-grass area for playing. In order to maintain the playground, it is necessary to know the area of the green-grass section and the area of the section with children's requisites. Try to calculate both of them.

The teacher encourages students to work on the problem and collaborate in order to find a solution of the problem. The teacher encourages students to use appropriate software to plot the circles described in the problem.</td></tr>
</table>

Previous knowledge assumed:	- analytic geometry, sketching figures in Dekart coordinate system - differentiation of functions with two real variables - algebraic equations - calculating double integrals	☐Homework **Assessment of learning** ☐Test ☐Quiz ☐Presentation ☐Project ☐Published work
Introduction / Theoretical basics	With the definition of the double integral, students are introduced with its application in calculating area, i.e. the value of the integral $\iint_D dxdy$ is equal to the area of the region $D \subseteq \mathbf{R}^2$. When D is circle or another shape constructed with circles, the integral above is easier to be calculated using polar coordinates, i.e substituting $x=\rho\cos\varphi$ and $y=\rho\sin\varphi$. If the pair (x,y) represents coordinates of a point within the circle $x^2+y^2=R^2$, such that the distance between the point and $(0,0)$ is ρ and the angle between that segment with length ρ and positive way of x-axes is φ, then according to the definition of trigonometric functions, $(x,y)=(\rho\cos\theta,\rho\sin\theta)$. For the points within the circle, the distance to $(0,0)$ can be maximum R, thus $0\le\rho\le R$. For the points within the circle for the angle φ holds: $0\le\varphi\le 2\pi$. If we have to calculate the integral $\iint_D dxdy$ over the region D: $x^2+y^2=R^2$, we have to calculate $\int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy$. If we make substitution with polar coordinates, we have to calculate the integral with rectangular region of integration: $\int_0^{2\pi} d\varphi \int_0^R \rho d\rho$ which is far easier and faster to calculate then the previous one. That is the advantage of introducing polar coordinates.	

	Teacher can solve students some examples using polar coordinates before solving the main problem set at the beginning of the lesson. Different software can be used to plot the lines that determine the region in the examples.	
Action	To solve the given problem, we can plot the circles that form the children's playground such that coordinates beginning $(0,0)$ is in the center of one of the circles and the segment between the centers of the circles lies on the x-axes. The equation for the first circle will be $x^2+y^2=64$ and for the second one it will be $(x-8)^2+y^2=64$.  If we write x and y in polar form, $x=\rho\cos\varphi$ and $y=\rho\sin\varphi$ and substitute it in the equations of both circles, we can conclude that that the distance ρ between each point within the second circle that is right of the intersection, satisfies $8\le\rho\le16\cos\varphi$. To determine the angle φ for those points, we determine first the intersection points of the two circles and plot lines between each intersection point and coordinates	

	beginning. One of the lines of $y = x\sqrt{3}$ and the other one is $y = -x\sqrt{3}$.  The angle between the line above the x-axes and x-axes is $\pi/3$ and the angle between the line bellow the x-axes and x-axes is $-\pi/3$. Thus, $-\pi/3 \le \varphi \le \pi/3$. Then the area within the second circle and right of the intersection is equal with the integral $\int_{-\pi/3}^{\pi/3} d\varphi \int_{8}^{16\cos\varphi} \rho d\rho$. As in the children's playground there is the same area left of the intersection, thus the area with the green grass will be $P = 2\int_{-\pi/3}^{\pi/3} d\varphi \int_{8}^{16\cos\varphi} \rho d\rho$. The area of the intersection (the area with the playing requisites in the playground) can be calculated as the difference of the circle area and the area left of the intersection, i.e. $P_1 = 64\pi - \int_{-\pi/3}^{\pi/3} d\varphi \int_{8}^{16\cos\varphi} \rho d\rho$.	

Materials / equipment / digital tools / software	Literature given in the references at the end of the document / Digital device which supports software for plotting graph or can be used to plot online Various online graph plotter or software for graph plotting (Geogebra or similar)	
Consolidation	With the given examples students can consider that double integrals are important for solving real life problems. Students will learn how to get calculating double integrals easier, with introducing polar coordinates. Students can use technology, different digital tools and software as a help for solving problems, but can also realize that even with technology, solving different everyday problems is difficult without math knowledge.	

Reflections and next steps	
Activities that worked	**Parts to be revisited**
Problem solving, collaboration, using technology	Depends on the students, in a conversation with students the teacher will realize the difficulties that students had and then revisit appropriate parts.

References

[1] E. Atanasova, S. Georgieva (2002), *Matematika 2*, Universitet “Sv. Kiril I Metodij” - Skopje
[2] S. Calaway D. Hoffman and D.Lippman (2014) “Applied Calculus”
[3] *P.D. Lax*, M. S.Terrell (2014) “Calculus with Applications”, Springer

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<table>
<tr><th colspan="3">TOPIC PLAN</th></tr>
<tr><td>Partner organization</td><td colspan="2">Goce Delcev University – Stip, North Macedonia</td></tr>
<tr><td>Topic</td><td colspan="2">Function with Two Variables: Application of Derivatives</td></tr>
<tr><td>Lesson title</td><td colspan="2">Minimizing and Maximizing Problems</td></tr>
<tr><td>Learning objectives</td><td>✓ Students will acquire and deal with derivatives of functions with several variables;
✓ Students will be able to estimate minimum and maximum of different sizes using differentiation of functions with two variables;
✓ Students will be able to deal with different problems in everyday life, which require calculating minimum or maximum value of a given size;
✓ Students are encouraged to use technology and different software in their work, while considering problem - based situations.</td><td rowspan="2">Strategies/Activities
☐Graphic Organizer
☐Think/Pair/Share
☐Modeling
☐Collaborative learning
☐Discussion questions
☐Project based learning
☐Problem based learning

Assessment for learning
☐Observations
☐Conversations
☐Work sample
☐Conference
☐Check list
☐Diagnostics

Assessment as learning
☐Self-assessment
☐Peer-assessment
☐Presentation</td></tr>
<tr><td>Aim of the lecture / Description of the practical problem</td><td>The aim of the lecture is to make students able to calculate derivatives of a function with several variables and apply the derivatives to calculate minimum and maximum of given size.

The teacher gives the next problems to the students:

1. We have to make a tin tank in a form of rectangular cuboid that will collect 125 liters liquid. Which dimensions of the tank will require the least amount of material for its construction?

2. We have to make box which requires 200 cm^2 cardboard for its construction. Which dimensions should the box be in order its volume to be the largest possible?

The teacher divides students into two groups and associates a problem to each group. Students have to work and collaborate in order to find a solution of the problem.</td></tr>
</table>

Previous knowledge assumed:

- derivatives of functions with one real variable
- area and volume of geometric 3D-forms
- algebraic equations
- differentiation of functions of two real variables
- determinants of second and third order

☐Graphic Organizer
☐Homework

Assessment of learning
☐Test
☐Quiz
☐Presentation
☐Project
☐Published work

Introduction / Theoretical basics

1. About the first problem, students have to consider that the volume of the cuboid will be 125 *l* =125 dm^3 and that the volume can be calculated with the formula *V=abc* where *a*, *b* and *c* are the dimensions of the cuboid. A sketch of the cuboid is necessary. They also have to consider that the amount of material needed for the tin tank construction is equal to the tank's area. The area can be calculated with the formula *P=2(ab+bc+ac)*. Thus, students have to minimize the area of the rectangular cuboid with fixed volume.

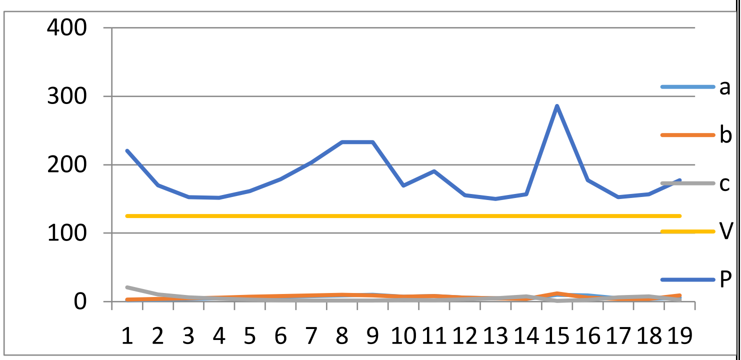
Students are encouraged to use Excel and formulas in it to consider how the area is changing with the change of cuboide's dimensions, while the volume is fixed.

One example for such Excel spreadsheet is following:

a	*b*	*c*	*V*	*P*
2	3	20,83333	125	220,3333333
3	4	10,41667	125	169,8333333
4	5	6,25	125	152,5
5	6	4,166667	125	151,6666667
6	7	2,97619	125	161,3809524
7	8	2,232143	125	178,9642857
8	9	1,736111	125	203,0277778
9	10	1,388889	125	232,7777778
10	9	1,388889	125	232,7777778
7	7	2,55102	125	169,4285714

8	8	1,953125	125	190,5
6	6	3,472222	125	155,3333333
5	**5**	**5**	**125**	**150**
4	4	7,8125	125	157
10	12	1,041667	125	285,8333333
9	6	2,314815	125	177,4444444
5	4	6,25	125	152,5
4	4	7,8125	125	157
6	9	2,314815	125	177,4444444

Students can create different charts with the data in the table, using Excel:



According to the values in the table, students can realize that the area is the smallest when three dimensions of the rectangular cubic are equal. But, this conclusion needs scientific support…

2. About the second problem, students' have to consider that the area of the box is 200 cm^2 and have to remind that we can calculate the area of the box with the formula *P=2(ab+bc+ac)* where *a*, *b* and *c* are the dimensions of the box. Students have to determine the dimensions of the box which gives the largest volume. The volume is calculated with the formula *V=abc*. Thus, students have to maximize the volume.

As well as in the first problem, students are encouraged to use digital tools in order to determine the solution easier. They can use Excel, too, for considering how the volume is changing with the change of the dimensions, while the area is fixed.

One example for such Excel spreadsheet is following:

a	***b***	***c***	***P***	***V***
5	5	7,5	200	187,5
6	6	5,333333	200	192
7	7	3,642857	200	178,5
6	7	4,461538	200	187,3846154
6,5	6,5	4,442308	200	187,6875
7,5	7,5	2,916667	200	164,0625
6,2	6,2	4,964516	200	190,836
6	5	6,363636	200	190,9090909
7	5	5,416667	200	189,5833333
6,2	5,7	5,433613	200	192,0238992
6,2	5,9	5,241322	200	191,7275702
6,1	6	5,239669	200	191,7719008
6	**5,9**	**5,428571**	**200**	**192,1714286**
9	7	2,3125	200	145,6875
10	5	3,333333	200	166,6666667
10	8	1,111111	200	88,88888889
9	8	1,647059	200	118,5882353
6,1	5,9	5,334167	200	191,9766583
6,1	5,8	5,430252	200	192,1223193

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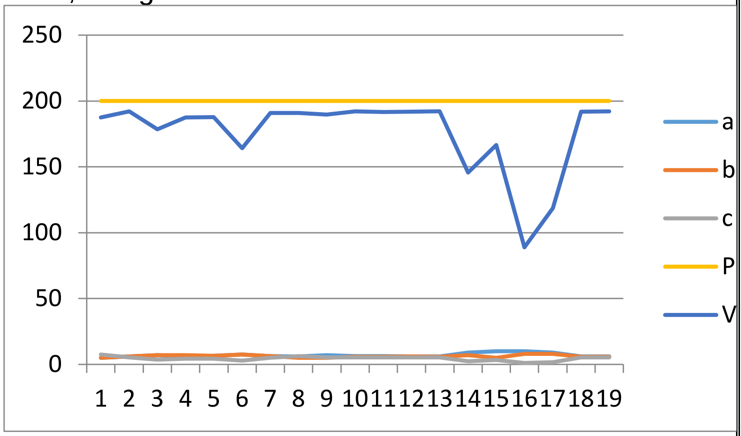
Students can create different charts with the data in the table, using Excel:



It is a little bit difficult for giving conclusion about the result in this problem, although it will be heuristically done.

Thus, mathematical support, algorithms and formulas are necessary for exact calculation of the solution in both of the problems.

Such minimizing and maximizing problems can easily be solved with an application of derivatives of functions with several variables. In these certain problems, we apply functions with two variables.

If $z = f(x, y)$ is given function with two independent variables, we determine its minimum/maximum values with the next algorithm: 1) we first calculate partial derivatives of first order, i.e. $z'_x = \frac{\partial z}{\partial x}$ and $z'_y = \frac{\partial z}{\partial y}$; 2) we solve the system of equations $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$.
The solutions of the system give us possible extreme values and are called stationary points; 3) we calculate the partial derivatives of the second order for the function $z = f(x, y)$, i.e. $z''_{xx} = \frac{\partial^2 z}{\partial x^2}$, $z''_{xy} = \frac{\partial^2 z}{\partial y \partial x}$, $z''_{yy} = \frac{\partial^2 z}{\partial y^2}$ and $z''_{yx} = \frac{\partial^2 z}{\partial x \partial y}$; 4) we calculate the value of each partial derivative of the second order in each stationary point. $z''_{xy} = z''_{yx}$ is necessary condition for

	the function to reach minimum/maximum value in certain stationary point. 5) we calculate the determinant $\Delta=\begin{vmatrix} z''_{xx}(x_0,y_0) & z''_{xy}(x_0,y_0) \\ z''_{yx}(x_0,y_0) & z''_{yy}(x_0,y_0) \end{vmatrix}$ for each stationary point (x_0,y_0); 6) if $\Delta<0$ in the considered stationary point, the function doesn't reach an extreme value in that point. If $\Delta=0$ we cannot conclude anything about the extreme value in that point and if $\Delta>0$ we conclude that the function has extreme value in the considered stationary point; 7) in order to be $\Delta>0$, because $z''_{xy}(x_0,y_0)=z''_{yx}(x_0,y_0)$, thus the sign of $z''_{xx}(x_0,y_0)$ and $z''_{yy}(x_0,y_0)$ must be the same. If $z''_{xx}(x_0,y_0)>0$ then the function reaches local minimum in the considered stationary point. If $z''_{xx}(x_0,y_0)<0$ then the function reaches local maximum in the considered stationary point. If we consider certain size as function with to variables, we can calculate its extreme values with previously described procedure.	
Action	For the first given problem, we know that $V=abc$ and $V=125$, thus $abc=125$, i.e. $c=\frac{125}{ab}$. We have to minimize the area, and the area is $P=2(ab+bc+ac)$. Substituting $c=\frac{125}{ab}$, we have $P=2\left(ab+b\cdot\frac{125}{ab}+a\cdot\frac{125}{ab}\right)$, i.e. $P=2\left(ab+\frac{125}{a}+\frac{125}{b}\right)$. Now we consider the area as a function with two real variables and following the algorithm above, we calculate that for $a=b=c=5$ we have minimal area. In a same way we can solve the second problem.	
Materials / equipment / digital tools / software	Literature given in the references at the end of the document / Digital device which supports Excel / Excel	

Consolidation	With the given examples students can consider that the real functions and their derivatives are important for solving real life problems. Students will learn how to calculate partial derivatives and how to apply differentiation and derivatives to maximize / minimize certain value by given conditions. Students can use technology, different digital tools and software as a help for solving problems, but can also realize that even with technology, solving different everyday problems is difficult without math knowledge.
Reflections and next steps	
Activities that worked	**Parts to be revisited**
Problem solving, collaboration, using technology	Depends on the students, in a conversation with students the teacher will realize the difficulties that students had and then revisit appropriate parts.
References	
[1] E. Atanasova, S. Georgieva (2002), *Matematika 2*, Universitet “Sv. Kiril I Metodij” - Skopje [2] S. Calaway D. Hoffman and D.Lippman (2014) “Applied Calculus” [3] *P.D. Lax*, M. S.Terrell (2014) “Calculus with Applications”, Springer	

Appendix 2. Tests for checking students' knowledge

APPLICATION OF DERIVATIVES -

MINIMIZING AND MAXIMIZING PROBLEMS

Test for checking students' knowledge

1. Calculate the first derivative of the next functions:

   a) $f(x)=x^3$ c) $f(x)=\sqrt[3]{x}$ e) $f(x)=\dfrac{3^x}{\sqrt[3]{x}}$

2. Calculate the first derivative of the next functions:

   a) $f(x)=(x+2)^4$ c) $f(x)=\sqrt{x^2-2x}$

3. Calculate the second derivative of the next functions:

   a) $f(x)=x^2(x-1)$ c) $f(x)=\left(1-\sqrt{x}\right)\sin x$

4. Calculate the local extreme values of the functions:

   a) $f(x)=\dfrac{x^3}{3}-\dfrac{x^2}{2}$ c) $f(x)=\dfrac{x-2}{x+1}$

5. Which two numbers with product 64 give the lowest sum?

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APPLICATION OF DERIVATIVES -

MINIMIZING AND MAXIMIZING PROBLEMS

Test for checking students' knowledge

1.Calculate the first derivative of the next functions:

b) $f(x)=\frac{1}{x^3}$ d) $f(x)=\frac{x^3}{\sqrt[3]{x}}$ f) $f(x)=2^x\cdot\ln x$

2.Calculate the first derivative of the next functions:

b) $f(x)=\cos(x+2)$ d) $f(x)=\frac{2}{(x-1)^3}$

3.Calculate the second derivative of the next functions:

b) $f(x)=(1+e^x)\ln x$ d) $f(x)=\frac{x^3}{3}-\frac{x^2}{2}$

4.Calculate the local extreme values of the functions:

b) $f(x)=2x^3-3x^2$ d) $f(x)=\frac{x-1}{x^2+2}$

5.Which two numbers with sum 100 give the greatest product?

ESTIMATING AREA WITH DOUBLE INTEGRALS USING POLAR COORDINATES

Test for checking students' knowledge

1. Sketch the area that is calculated with the integral $\int_{-1}^{1} dx \int_{3}^{4} dy$.

2. Sketch the area that is calculated with the integral $\int_{-1}^{1} dx \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dy$, and then calculate the area transforming it into square (using polar coordinates).

3. Calculate the area which is intersection of the circles $x^2 + y^2 = 4$ and $x^2 + (y-1)^2 = 4$

FUNCTIONS WITH TWO VARIABLES – APPLICATION OF PARTIAL DERIVATIVES

Test for checking students' knowledge

1.Calculate partial derivatives of first and second order for the next functions:

c) $z = x \ln 2y$ b) $z = x^2 \sqrt{y}$ c) $z = \sqrt[3]{xy}$

2.Calculate partial derivatives of first and second order for the next functions:

a) $z = \ln \frac{2x}{3y}$ b) $z = x + y + 4 + 4 \sin x \sin y$

3.Determine the extreme values of the next functions:

c) $f(x, y) = \frac{x^2}{2} + \frac{y^3}{3} + 3xy$ b) $z = x^2 - xy + y^2 - 2x - 2y$

4.Determine the dimensions of cuboid with volume 521 liters, such that it will have as small as possible area.

Short test for the lesson

“Definition and basic notions for Ordinary Differential Equations (ODE)”

Circle the correct answer:

1. The differential equation $y'' = x$ is:
   a) PDE; b) ODE.

2. The differential equation $y'' = x$ is of order:
   a) 3; b) 1; c) 2.

3. The general solution of the differential equation $y'' = x$ contain:
   a) one integration constant;
   b) two integration constants;
   c) three integration constants.

4. The solution $y = \frac{x^2}{2} + C$ of the differential equation $y' = x$ is:
   a) a general solution; b) a particular solution.

5. The solution $y = \frac{x^2}{2}$ of the differential equation $y' = x$ is:
   a) a general solution; b) a particular solution.

Answer the questions:

1. Definition for ODE.

2. What is the difference between the general and the particular solution of the differential equation? Explain geometric visualization of the general and the particular solution for a differential equation!

3. What are the steps for the solving of the task from applied mathematics, natural sciences, and technology?

Short test for the lesson

“Ordinary Differential Equations of first order”

Circle the correct answer:

1. Which type is the differential equation $y' = f(\frac{y}{x})$?

a) Differential equation with separable variables;
b) Homogeneous differential equation

2. The general solution of the differential equation $y' = x$ contains:

a) one integration constant;
b) two integration constants;
c) three integration constants.

3. Which type is the differential equation $y' = x$?

a) Differential equation with separable variables;
b) Homogeneous differential equation

4. The solution $y = \frac{x^2}{2}$ of the differential equation $y' = x$ is:

a) general solution; b) particular solution.

Answer the questions:

5. Definition of differential equation of first order.

6. What is the difference between the general and the particular solution of the differential equation? Explain geometric visualization of the general and the particular solution for a differential equation!