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• Use a line integral to compute the work done in moving an object along a curve in a vector field.
• Describe the circulation of a vector field.</td><td rowspan="3">Strategies/Activities
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<tr><td>Aim of the lecture / Description of the practical problem</td><td>Practical problem: How would we compute the work done by vector field $\vec{F}$ in moving a particle along C?
• We need a vector line integral.</td></tr>
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• Parametrization of a space curve</td></tr>
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Introduction / Theoretical basics	The second type of line integrals are vector line integrals, in which we integrate along a curve through a vector field. In the following, let $\vec{F}(x,y,z) = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$, be a continued vector field in $\mathbb{R}^3$ that represents a force on a particle, and let C be a smooth curve in $\mathbb{R}^3$ contained in the domain of $\vec{F}$. **Definition.** Let f be a function with a domain that includes the smooth curve C that is parameterized by $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}, \quad a \le t \le b.$ The vector line integral of $\vec{F}$ along C is: $$\int_C \vec{F} \cdot d\vec{r}(t) = \int_a^b \vec{F}(\vec{r}(t))\vec{r}\,'(t)dt\,.$$ In this notation we have: $$\vec{r}\,'(t) = x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k}.$$ Therefore, the work done by $\vec{F}$ in moving the particle in the positive direction along C is defined as: $$W = \int_C \vec{F} \cdot d\vec{r}(t)\,.$$	☑Self-assessment ☐Peer-assessment ☐Presentation ☐Graphic Organizer ☐Homework **Assessment of learning** ☑Test ☑Quiz ☐Presentation ☐Project ☐Published work
Action	Questions to students: • How to calculate vector line integrals? • What are the properties of the vector line integrals? • Is any application? Concerning the evaluation of the vector line integrals we give the following examples. **Example.** Find the value of the vector line integral $$\int_C \vec{F} \cdot d\vec{r}(t)$$ where C is the circle parameterized with: $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j}\,, \quad 0 \le t \le \pi$	

and

$$\vec{F} = -y\vec{i} + x\vec{j}\,.$$

Note that in this example the vector field is two dimensional. Now we have:

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r}(t) &= \int_a^b \vec{F}(\vec{r}(t))\vec{r}\,'(t)dt \\ &= \int_0^\pi (-\sin t\vec{i} + \cos t\vec{j})\cdot(-\sin t\vec{i} + \cos t\vec{j})dt \\ &= \int_0^\pi (\sin^2 t + \cos^2 t)dt = \int_0^\pi dt = \pi.\end{aligned}$$

Properties. The vector line integral has the following properties:

$$\begin{aligned}\int_{-C} \vec{F}\cdot d\vec{r}(t) &= -\int_C \vec{F}\cdot d\vec{r}(t)\,; \\ \int_C (\vec{F}+\vec{G})\cdot d\vec{r}(t) &= \int_C \vec{F}\cdot d\vec{r}(t) + \int_C \vec{G}\cdot d\vec{r}(t) \\ \int_C k\vec{F}\cdot d\vec{r}(t) &= k\int_C \vec{F}\cdot d\vec{r}(t)\,, \quad \text{for a constant } \ k\,.\end{aligned}$$

Similar as in the scalar line integrals, the evaluation of the vector line integral over a piecewise smoot curve would be:

$$\int_C \vec{F}\cdot d\vec{r}(t) = \int_{C_1} \vec{F}\cdot d\vec{r}(t) + \int_{C_2} \vec{F}\cdot d\vec{r}(t)\,.$$

Another standard notation for the vector line integral is the following:

$$\int_C \vec{F}\cdot d\vec{r}(t) = \int_C Mdx + Ndy + Pdz\,.$$

Example. Find

$$\int_C x^2 ydx + (y-z)dy + xzdz$$

where:

a) C is the curve given with the equation:

$$\vec{r}(t) = \vec{i} + t^2\vec{j} + t\vec{k}\,, \quad 0 \le t \le 2;$$

b) C is the line between (1,0,0) and (1,4,2).

For

$$\vec{r}(t) = \vec{i} + t^2\vec{j} + t\vec{k}$$

we have

$$x(t) = 1,\, y(t) = t^2 \text{ i } z(t) = t$$

and,

$$x^2ydx + (y - z)dy + xzdz =$$

$$= 1^2t^2 \cdot 0 \cdot dt + (t^2 - t)2tdt + 1 \cdot tdt\,.$$

For the line integral we have:

$$\int_C x^2ydx + (y - z)dy + xzdz =$$

$$= \int_0^2 (2t^3 - 2t^2 + t)dt = \frac{14}{3}\,.$$

The line passing from (1,0,0) to (1,4,2) has the parametric equations:

$$x(t) = 1,\, y(t) = 4t \text{ and } z(t) = 2t$$

Where $0 \le t \le 1$ and we have:

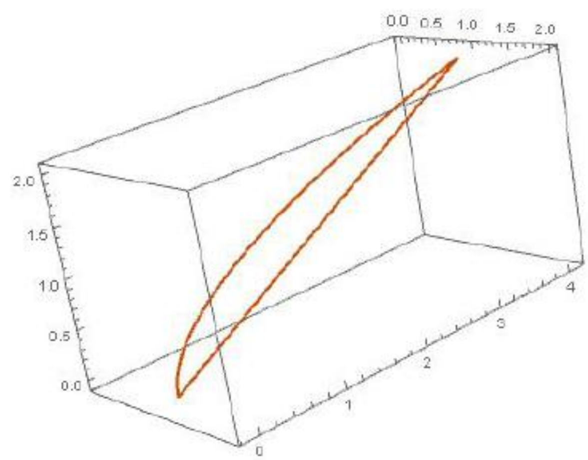
$$x^2ydx + (y - z)dy + xzdz =$$

$$= 1^2 4t \cdot 0dt + (4t - 2t)4dt + 2t \cdot 2dt\,.$$

For the line integral we got:

$$\int_C x^2y\,dx + (y-z)dy + xz\,dz = \int_0^1 12t\,dt = 6\,.$$

Note. Both curves are given in the following figure. They start at the same point and end in the same point. But the value of the integral is not the same.



Application.
Question to the students:

- Are you familiar with any application of the vector line integral?

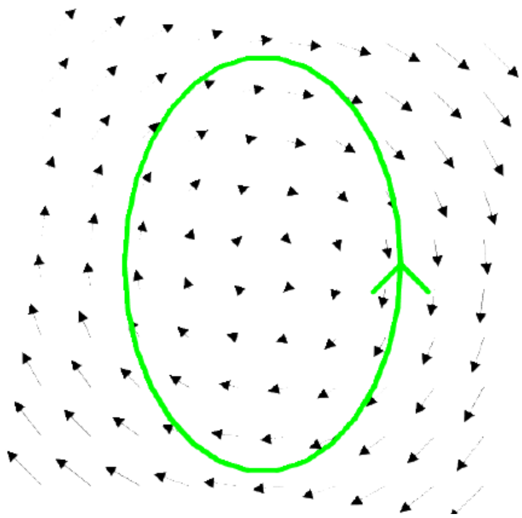
Discussion with the students.
Vector line integrals are extremely useful in physics. They can be used to calculate the work done on a particle as it moves through a force field, or the flow rate of a fluid across a curve.
In the following example we calculate the work done by a force using a vector line integral.

A work done by a force.
Example. How much work is required to move an object in vector force field

$$\vec{F} = yz\vec{i} + xy\vec{j} + xz\vec{k}$$

along the path:

	$\vec{r}(t) = t^2\vec{i} + t\vec{j} + t^4\vec{k},\ 0 \le t \le 1\,.$ We have: $$W = \int_C \vec{F} \cdot d\vec{r}(t)$$ $$= \int_0^1 (yz\vec{i} + xy\vec{j} + xz\vec{k}) \cdot (2t\vec{i} + \vec{j} + 4t^3\vec{k})dt$$ $$= \int_0^1 (t^5 \cdot 2t + t^3 + t^6 \cdot 4t^3)dt = \frac{131}{140}\,.$$ **Line integrals as circulation.** The vector line integral explains how the line integral of a vector field $\vec{F}$ over an oriented curve C "adds up" the component of the vector field that is tangent to the curve. In this sense, the line integral measures how much the vector field is aligned with the curve. If the curve C is a closed curve, then the line integral indicates how much the vector field tends to circulate around the curve C: In fact, for an oriented closed curve C; we call the line integral the "circulation" of $\vec{F}$ around C, see the following figure: $$\oint_C \vec{F} \cdot d\vec{r}(t) = \text{ circulation of } \vec{F} \text{ over } C\,.$$	

	 **Example.** Find the circulation of the vector field $$\vec{F} = y\vec{i} - x\vec{j}$$ over the ellipse $$\frac{x^2}{4} + \frac{y^2}{9} = 1\,.$$ First we represent the ellipse with its parametric equations: $$\vec{r}(t) = 2\cos t\vec{i} + 3\sin t\vec{j}\ \ 0 \leq t \leq 2\pi\,.$$ Next, we have: $$\oint_C \vec{F}\cdot d\vec{r}(t) \ \ =$$	

	$$= \int_0^{2\pi} (y\vec{i} - x\vec{j}) \cdot (-2\sin t\vec{i} + 3\cos t\vec{j})dt =$$ $$= \int_0^{2\pi} (3\sin t\vec{i} - 2\cos t\vec{j}) \cdot (-2\sin t\vec{i} + 3\cos t\vec{j})dt$$ $$= \int_0^{2\pi} (-6\sin^2 t - 6\cos^t) = -12\pi\,.$$	
Materials / equipment / digital tools / software	*The materials for learning* are given as a part of references of the end from this topic plan; *Equipment*: classroom, green board, chalk in different colours; *Digital tools*: laptop, projector, smart board; *Software*: Mathematica.	
Consolidation	• Use of materials, equipment, digital tools, software by teachers and students; • The teacher's discussion with the students through appropriate questions; • Independent solving of simple tasks by the students under the supervision of the teacher; • Given of examples by the teacher for introducing a new concept in a cooperation and a discussion with the students; • Assignment of homework by the teacher with a time limit until the next class.	

Reflections and next steps	
Activities that worked	**Parts to be revisited**
After the class, the teacher according to his personal perceptions regarding the success of the class fills in this part.	Through the success of the homework done by the students, questions and discussion at the beginning of the next class, the teacher comes to the conclusion which parts of this class should be revised.

References
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