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<table>
<tr><th colspan="3">TOPIC PLAN</th></tr>
<tr><td>Partner organization</td><td colspan="2">Sojuz na istrazuvaci na Makedonija-SIM Skopje</td></tr>
<tr><td>Topic</td><td colspan="2">Line integrals</td></tr>
<tr><td>Lesson title</td><td colspan="2">Scalar Line integrals</td></tr>
<tr><td>Learning objectives</td><td>• Calculate a scalar line integral along a curve.
• Use a line integral to compute the length of a curve.
• Use a line integral to calculate the mass of a wire.</td><td rowspan="3">Strategies/Activities
☐Graphic Organizer
☐Think/Pair/Share
☑ Modeling
☑ Collaborative learning
☑ Discussion questions
☐Project based learning
☑Problem based learning

Assessment for learning
☑Observations
☑Conversations
☑Work sample
☐Conference
☐Check list
☐Diagnostics

Assessment as learning</td></tr>
<tr><td>Aim of the lecture / Description of the practical problem</td><td>Practical problem: Calculate the mass of a spring in the shape of a curve parameterized by
$\vec{r}(t) = t\vec{i} + 2\cos t\vec{j} + 2\sin t\vec{k}$
$0 \leq t \leq \frac{\pi}{2}$, with a density function given by
$\rho(x, y, z) = e^x + yz$.</td></tr>
<tr><td>Previous knowledge assumed:</td><td>• Evaluation of integrals
• Parametrization of a plane curve
• Parametrization of a space curve</td></tr>
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Introduction / Theoretical basics

Scalar line integrals.
This type of integral we also call it line integral of f with respect to arc length.
For a formal description of a scalar line integral, let C be a smooth curve in space given by the parameterization:

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}, \quad a \le t \le b\,.$$

Let

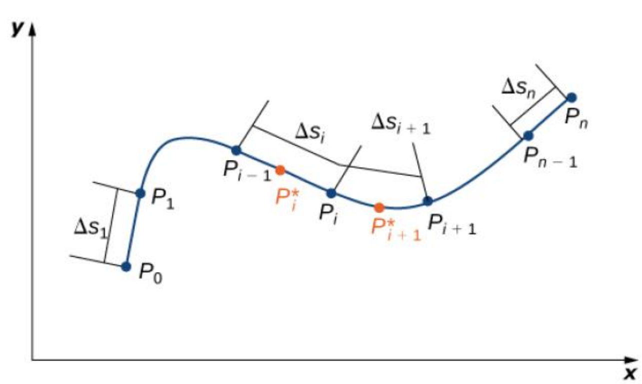
$$f = f(x, y, z)$$

be a function with a domain that includes curve C: To define the integral we begin as most definitions of an integral begin: we chop the curve into small pieces.

Partition the parameter interval [a,b] into n subintervals $[t_i, t_{i+1}]$ where $t_0=a$ and $t_n=b$. Let t_i^* be a value in the subinterval $[t_i, t_{i+1}]$. We denote the endpoints

$$\vec{r}(t_0), \vec{r}(t_1), \ldots, \vec{r}(t_n)$$

with $P_0, P_1, \ldots P_n$, see the figure below.



Points P_i divide the curve C into pieces $C_0, C_1, \ldots C_n$, with length:

$$\triangle s_0, \triangle s_1, \ldots, \triangle s_n.$$

At the end, we evaluate the function for the point P_i^*, multiply with Δs_i and sum for $1 \le i \le n$.

Definition. Let f be a function with a domain that includes the smooth curve C that is parameterized by

☑Self-assessment
☐Peer-assessment
☐Presentation
☐Graphic Organizer
☐Homework

Assessment of learning
☑Test
☑Quiz
☐Presentation
☐Project
☐Published work

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<td>$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}, \quad a \le t \le b.$
The scalar line integral of f along C is:
$$\int_C f(x,y,z)ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(P_i^*)\Delta s_i\,,$$
if this limit exists, and P_i^* and Δs_i are defined as before.</td>
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<td>Action</td>
<td>Questions to students:
How to calculate the new defined integrals?
How can we connect the length of a given integral with the scalar line integral over the same curve?
The question about how to evaluate the scalar line integral gives the following theorem.
Theorem. Let f be a continuous function with a domain that includes the smooth curve C with parameterization:
$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}, \quad a \le t \le b.$
Then
$$\int_C f(x,y,z)ds =$$
$$\int_a^b f(x(t),y(t),z(t))\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\;dt\,.$$
Example. Find
$$\int_C (x^2+y^2+z^2)ds$$
where C is given by x=cost, y=sint, z=t for $0 \le t \le 2\pi$ from (1,0,0) to (1,0,2π).
The curve we are about to integrate on is given on the following figure:</td>
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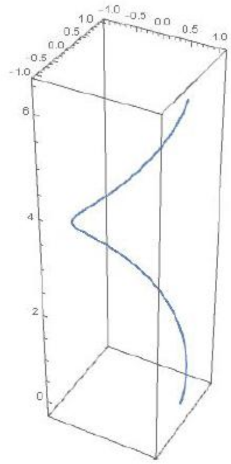

Using the theorem for the evaluation of the line integral we have:

$$\int_C f(x,y,z)ds =$$

$$= \int_0^{2\pi} (\cos^2 t + \sin^2 t + t^2)\sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}\ dt$$

$$= \int_0^{2\pi} (1+t^2)\sqrt{2}\,dt = 2\sqrt{2}\pi(1+4\pi^2/3)\,.$$

Next, if we want just to evaluate the length of the curve C, i.e. to calculate the arc length, we have to take:

$$f(x,y,z) = 1$$

and then the arc length of C will be evaluated by:

$$\int_C ds\,.$$

Example. Find the length of a wire with a parametrization: x=3t+1, y=4-2t, z=5+2t, 0≤t≤4.
We have:

$$\int_C ds = \int_0^4 \sqrt{(3)^2+(-2)^2+2^2}\ dt$$

$$= \int_0^4 \sqrt{17}\,dt = 4\sqrt{17}\,.$$

Properties. The scalar line integrals have the following properties:

$$\int_C kf(x,y,z)ds = k\int_C f(x,y,z)ds\,,$$

$$\int_C (f\pm g)(x,y,z)ds = \int_C f(x,y,z)ds \pm \int_C g(x,y,z)ds$$

Since $\Delta s_i>0$, when we swich the direction of the curve, in the line integral will not change.

$$\int_C f(x,y,z)ds = \int_{-C} f(x,y,z)ds\,.$$

Evaluation of the integral over a piecewise smooth curve is a simple thing to do. We evaluate the integral over each pieces, and then add them up. We have:

$$\int_C f(x,y,z)ds = \int_{C_1} f(x,y,z)ds + \int_{C_2} f(x,y,z)ds\,.$$

Application.

Question to the students:

- Are you familiar with any application of the scalar line integral?

Discussion with the students.

They can be used to calculate the length or mass of a wire, the surface area of a sheet of a given height, or the electric potential of a charged wire given a linear charge density.

A mass of a wire.

Suppose that a piece of wire is modeled by curve C in space. The mass per unit length (the linear density) of the wire is a continuous function

$$\rho(x,y,z).$$

We can calculate the total mass of the wire using the scalar line integral

$$\int_C \rho(x,y,z)ds\,.$$

Now, at the end of this class we give the solution of the opening problem calculating the mass of a spring in the shape of a parameterized curve.

We have:

	$$\int_C \rho(x,y,z)ds =$$ $$\int_0^{\pi/2} (e^x + yz)\sqrt{x'^2(t) + y'^2(t) + z'^2(t)}dt =$$ $$\int_0^{\pi/2} (e^t + 4\sin t\cos t)\sqrt{1^2 + (-2\sin t)^2 + (2\cos t)^2}dt =$$ $$\sqrt{5}\int_0^{\pi/2} (e^t + 4\sin t\cos t)dt = \sqrt{5}(e^{\pi/2} + 1)\,.$$	
Materials / equipment / digital tools / software	*The materials for learning* are given as a part of references of the end from this topic plan; *Equipment*: classroom, green board, chalk in different colours; *Digital tools*: laptop, projector, smart board; *Software*: Mathematica.	
Consolidation	• Use of materials, equipment, digital tools, software by teachers and students; • The teacher's discussion with the students through appropriate questions; • Independent solving of simple tasks by the students under the supervision of the teacher; • Given of examples by the teacher for introducing a new concept in a cooperation and a discussion with the students; • Assignment of homework by the teacher with a time limit until the next class.	

Reflections and next steps	
Activities that worked	**Parts to be revisited**
After the class, the teacher according to his personal perceptions regarding the success of the class fills in this part.	Through the success of the homework done by the students, questions and discussion at the beginning of the next class, the teacher comes to the conclusion which parts of this class should be revised.
References	

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[3] T. M. Apostol: Vector analysis, line integrals, and surface integrals, 1960, California Institute of Technology.
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[6] https://math.libretexts.org/

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