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Programme: Erasmus+
Key Action: Cooperation for innovation and the exchange of good practices
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Ref. No.: 2020-1-RS01-KA203-065388

**PROJECT TITLE: Mathematics of the Future: Understanding and Application of Mathematics with the help of Technology**

**Intellectual Output 3: Development of new STEAM methodology and technology**

**Result-IO3- 3.4 : 3.4 Exemplary Tests**

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Programme: Erasmus+
Key Action: Cooperation for innovation and the exchange of good practices
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| Ref. No.: | 2020-1-RS01-KA203-065388 |

# 1. Test-Functions – UNS

**Test- Examining functions without derivatives**

1. The functions $f$ are given in the table below

| | | |
|---|---|---|
| $f(x)=x$ | $f(x)=x^2$ | $f(x)=2^x$ |
| $f(x)=\left(\frac{1}{3}\right)^x$ | $f(x)=\log_2 x$ | $f(x)=\log_{1/3} x$ |
| $f(x)=\ln x$ | $f(x)=\sin x$ | $f(x)=\cos x$ |
| $f(x)=\operatorname{tg} x$ | $f(x)=\operatorname{ctg} x$ | $f(x)=\frac{1}{x}$ |

Sketch the graphs and examine the properties of the functions:

1) $f(x)+a$, $f(x+a)$, $af(x)$ и $f(ax)$,
2) $f(x+a)+b$, $af(x)+b$, $f(ax)+b$ и $f(ax+b)$,

for different values of the parameters $a$ and $b$, by using package *GeoGebra*.

Programme: Erasmus+
Key Action: Cooperation for innovation and the exchange of good practices
Action Type: Strategic Partnerships for higher education
Ref. No.: 2020-1-RS01-KA203-065388

## 2. Test-Definite integral--UNS

**Test- Functions given by integral**

The function $F$, is given as $F(t) = \int_{-2}^{t} f(x)dx$ , where $f$ is function defined and continuous on interval $[-2,8]$. The function $f$ is given with its graph shown in Figure. It holds $P_1 = P_2 = 2.5$.

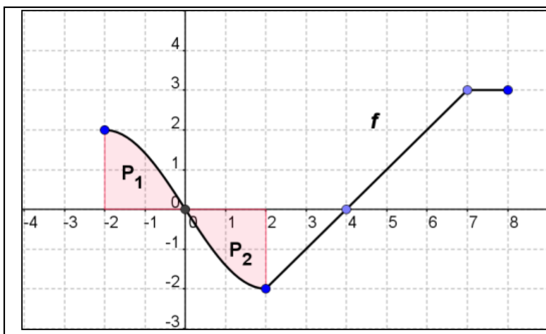



Figure 1.

1. Fulfill the following table

| $t$ | -2 | -1 | 1 | 1 | 2 | 3 | 4 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|---|
| $F(t)$ | | | | | | | | | | |

Programme: Erasmus+
Key Action: Cooperation for innovation and the exchange of good practices
Action Type: Strategic Partnerships for higher education
Ref. No.: 2020-1-RS01-KA203-065388

2. Determine the domain of $F$ .
3. Determine the range of $F$ .
4. Determine the intervals where the funkcija $F$ increases. Determine the intervals where the funkcija $F$ decreases.
5. Determine the values of $t$ in which the $F$ has local minimum and local maximum.
6. Determine the intervals where the funkcion $F$ convex. Determine the intervals where the funkcion $F$ concave.
7. Determine the sadle points of the function $F$ .
8. Connect the functions $f$ i $F$ ?
9. Draw the points $(t, F(t))$ , and sketch the graph of $F$.
10. Determine:
    a) $\int_0^2 f(x)dx$ , $\int_{-2}^2 f(x)dx$, $\int_2^4 f(x)dx$ , $\int_2^5 f(x)dx$ , $\int_7^8 f(x)dx$, (the function $f$ is given by its graph in Figure 1.

Programme: Erasmus+
Key Action: Cooperation for innovation and the exchange of good practices
Action Type: Strategic Partnerships for higher education
Ref. No.: 2020-1-RS01-KA203-065388

## 3. Test - Derivatives--UNS

### Test - partial derivatives

1. For $u = x^2y^3 + 5z^4 - 2xy$ , find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$.
2. For $f(x, y) = 8x^2y^3 - \frac{x}{y}$ find $f_x$ and $f_y$.
3. A publisher's production function for textbooks is given by $p(x, y) = 72x^{0.8}y^{0.2}$, where $p$ is the number of books produced, $x$ is units of labor, and $y$ is units of capital. Determine the marginal productivities at $x = 90$ and $y = 50$.
4. Find the equation of the tangent plane to $z = x^2 + y^2$ at $(0,0)$.
5. Find the linear approximation to $z = xy + x^2$ at $(1,0)$.

Programme: Erasmus+
Key Action: Cooperation for innovation and the exchange of good practices
Action Type: Strategic Partnerships for higher education
Ref. No.: 2020-1-RS01-KA203-065388

## 4. Test –Vector fiels--SIM

**Topics: Vector fields, Line integrals and Surface integrals**

1. Find the integral

$$\int_C \vec{F} \cdot d\vec{r}$$

   where

$$\vec{F} = (x^2 - y)\vec{i} + 2x\vec{j}$$

   and C is the curve (parabola) $y=1-x^2$ from A(-1,0) to B(1,0).
2. Find the integral

$$\int_C \vec{F} \cdot d\vec{r}$$

   where $\vec{F}$ is the vector field

$$\vec{F} = (x+y)\vec{i} + xy\vec{j}$$

   and C is the broken line between the points A(0,0), B(0,2) and C(1,2).
3. Determine the constant a, so the vector field

Programme: Erasmus+
Key Action: Cooperation for innovation and the exchange of good practices
Action Type: Strategic Partnerships for higher education
Ref. No.: 2020-1-RS01-KA203-065388

$$\vec{F} = (y^2 + 2x)\vec{i} + axy\vec{j}$$

has its potential.

4. Show that

$$\oint_C 4x^3ydx + x^4dy = 0$$

for every closed curve C.

5. For what values of a and b will

$$\vec{F} = yz^2\vec{i} + (xz^2 + ayz)\vec{j} + (bxyz + y^2)\vec{k}$$

is a be a conservative vector field? Use these values and find the corresponding potential function.

6. Let S be that portion of the plane -12x+4y+3z=12 projecting vertically onto the region $(x-1)^2 + y^2 \leq 4$.
Evaluate

$$\iint_S z d\sigma .$$

7. Find the flux

$$\iint_S \vec{F} \cdot d\vec{S}$$

trough the portion of the plane x+y+z=1 lying in the first octant if

$$\vec{F} = \vec{i}$$

.

8. Find the flux

$$\iint_S \vec{F} \cdot d\vec{S}$$

of the vector field

$$\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$$

Trough the central sphere with radius 2.

## 5. Test -BigO–BMU

Big O

Programme: Erasmus+
Key Action: Cooperation for innovation and the exchange of good practices
Action Type: Strategic Partnerships for higher education
Ref. No.: 2020-1-RS01-KA203-065388

**Exercises**

1. Assume that each of the expressions below gives the processing time $T(n)$ spent by an algorithm for solving a problem of size $n$. Select the dominant term(s) having the steepest increase in $n$ and specify the lowest Big-$O$ complexity of each algorithm.

| Expression | Dominant term | $O(n)$ |
|---|---|---|
| $5n + 0.001n^3 + 0.025n$ | | |
| $500n + 100n^{1.5} + 50n \log_{10} n$ | | |
| $0.3n + 5n^{1.5} + 2.5n^{1.75}$ | | |
| $n^2 \log_2 n + n(\log_2 n)^2$ | | |
| $n \log_2 n + n \log_3 n$ | | |
| $3\log_8 n + \log_2 \log_2 \log_2 n$ | | |
| $100n + 0.01n^2$ | | |
| $0.01n + 100n^2$ | | |
| $2n + n^{0.5} + 0.5n^{1.25}$ | | |
| $0.01n \log_2 n + n(\log_2 n)^2$ | | |
| $100n \log_3 n + n^3 + 100n$ | | |
| $0.003\log_4 n + \log_2 \log_2 n$ | | |

2. The statements below show some features of Big-$O$ notation for the functions $f = f(n)$ and $g = g(n)$. Determine whether each statement is TRUE or FALSE and correct the formula in the latter case.

| Expression | Is it TRUE or FALSE? | If it is FALSE then the correct formula |
|---|---|---|
| Rule of sums: $O(f + g) = O(f) + O(g)$ | | |
| Rule of products: $O(f \cdot g) = O(f) \cdot O(g)$ | | |

Programme: Erasmus+
Key Action: Cooperation for innovation and the exchange of good practices
Action Type: Strategic Partnerships for higher education
Ref. No.: 2020-1-RS01-KA203-065388

| | | |
|---|---|---|
| Transitivity: if $g = O(f)$ and $h = O(f)$ then $g = O(h)$ | | |
| $5n + 8n^2 + 100n^3 = O(n^4)$ | | |
| $5n + 8n^2 + 100n^3 = O(n^2 \log_2 n)$ | | |

3. Algorithms $A$ and $B$ spend exactly $T_A(n) = 0.1n^2 \log_{10} n$ and $T_A(n) = 2.5n^2$ microseconds, respectively, for a problem of size $n$. Choose the algorithm, which is better in the Big-$O$ sense, and find out a problem size $n_0$ such that for any larger size $n > n_0$ the chosen algorithm outperforms the other. If your problems are of the size $n \leq 10^9$, which algorithm will you recommend to use?

4. Algorithms $A$ and $B$ spend exactly $T_A(n) = 5n \log_{10} n$ and $T_B(n) = 25n$ microseconds, respectively, for a problem of size $n$. Which algorithm is better in the Big-$O$ sense? For which problem sizes does it outperform the other?

## 6. Test - Improper integral—UPT

Programme: Erasmus+
Key Action: Cooperation for innovation and the exchange of good practices
Action Type: Strategic Partnerships for higher education
Ref. No.: 2020-1-RS01-KA203-065388

**Test- Improper Integrals**

1. We consider the functions $f$ given in the table below together with the corresponding intervals:

| | | |
|---|---|---|
| $f(x)=\frac{1}{x}, x\in(1,\infty)$ | $f(x)=\frac{1}{x^2}, x\in(1,\infty)$ | $f(x)=\frac{1}{x-2}, x\in(2,4)$ |
| $f(x)=\left(\frac{1}{3-x}\right)^2, x\in(1,3)$ | $f(x)=e^{-5x}, x\in(0,\infty)$ | $f(x)=\frac{x}{\sqrt{4-x}}, x\in(1,4)$ |
| $f(x)=\frac{1}{\sqrt{x+2}}, x\in(-2,2)$ | $f(x)=\frac{x+2}{x^3-1}, x\in(1,\infty)$ | $f(x)=\frac{x-1}{x+3}, x\in(2,3)$ |

In each case we consider $\int_a^b f(x)dx$ ,where (a,b) is the corresponding interval from the table.

i) Which of the above integrals are improper integrals and why?

ii) Which of the above integrals are convergent?

iii) If the integral is convergent compute its value.

2. Compute $\int_{-2}^{2}\frac{1}{\sqrt{x+2}}dx$.

3. Prove that the following improper integrals are divergent:

i) $\int_0^\infty \cos(x)dx$; ii) $\int_0^\infty x\sin(x)dx$;

Programme: Erasmus+
Key Action: Cooperation for innovation and the exchange of good practices
Action Type: Strategic Partnerships for higher education
Ref. No.: 2020-1-RS01-KA203-065388

4. Study the convergence of the improper integrals using the comparison test:

i) $\int_1^\infty \frac{\arctan(x)}{x^2}dx$; ii) $\int_1^\infty \frac{\arctan(x)}{x}dx$; iii) $\int_4^8 \frac{1}{\sqrt{x-4}\cdot\sqrt[5]{8-x}}dx$

5. Study the convergence of following integrals, and if it is possible compute their values:

i) $\int_1^\infty \frac{1}{x(x^{2022}+1)}dx$; ii) $\int_{-\infty}^{-2} \frac{2x+1}{(x^2+1)(x+1)}dx$; iii) $\int_0^1 \frac{\sin x}{1-x^2}dx$.

Programme: Erasmus+
Key Action: Cooperation for innovation and the exchange of good practices
Action Type: Strategic Partnerships for higher education
Ref. No.: 2020-1-RS01-KA203-065388

## 7. Test- Complex numbers –UNS

1. Fill in the following table:

| $n$ | 0 | 4 | 10 | 41 | 62 | 403 | 2022 |
|---|---|---|---|---|---|---|---|
| $i^n$ | | | | | | | |

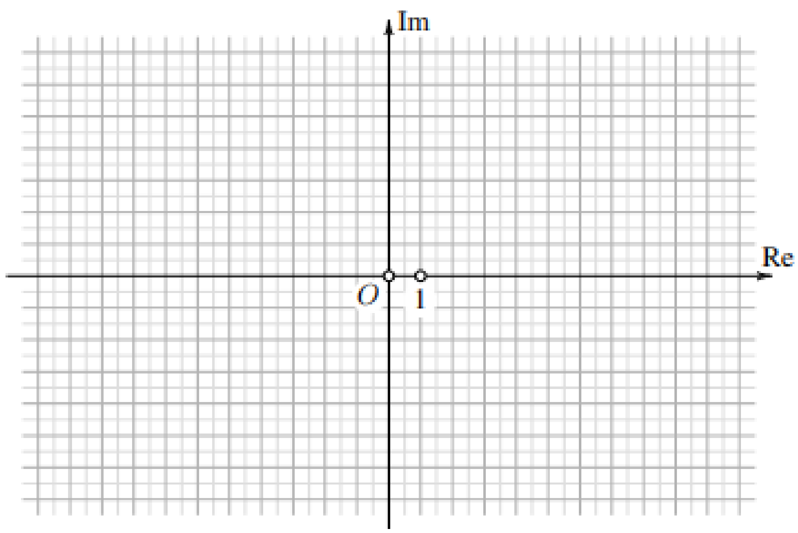
2. Compute the real and the imaginary part of the following complex numbers:

$$-12+13i, \quad i, \quad -2, \quad 3i-5, \quad 2i+5i^2.$$

3. Compute the conjugate of each complex numbers from the exercise above.

4. Graph the following complex numbers in the complex plane and for each of them compute the modulus and the tangent of the argument $\tan(\varphi)$, or compute the argument directly if the tanget is not defined:

$$a=3+4i, \quad b=-5+5i, \quad c=4, \quad d=5i, \quad e=9-4i.$$



Programme: Erasmus+
Key Action: Cooperation for innovation and the exchange of good practices
Action Type: Strategic Partnerships for higher education
Ref. No.: 2020-1-RS01-KA203-065388

5. Compute by hand:

$$\frac{1+i}{1-i},\quad \frac{1}{1-\frac{1}{i}},\quad \frac{(4-7i)(5+3i)}{(7i-4)(3i-5)}$$

6. Write SageMath commands that compute the expressions in the exercise above.

7. Using complex arithmetic in SageMath write a Python program that draws a regular 9-gon.

8. Write SageMath commands that solve the following equation:

$$3x^3-2x^2+4x-1=0.$$

## 8. GDU-TEST-ODE

Short test for the lesson
“Definition and basic notions for Ordinary Differential Equations (ODE)”

### Circle the correct answer

1. The differential equation $y''=x$ is:

a) PDE; b) ODE.

2. The differential equation $y''=x$ is of order:
   a) 3; b) 1; c) 2.

3. The general solution of the differential equation $y''=x$ contain:
   a) one integration constant;
   b) two integration constants;
   c) three integration constants.

4. The solution $y=\frac{x^2}{2}+C$ of the differential equation $y'=x$ is:

a) a general solution; b) a particular solution.

5. The solution $y = \frac{x^2}{2}$ of the differential equation $y' = x$ is:

a) a general solution; b) a particular solution.

**Answer the questions:**

1. Definition for ODE.

2. What is the difference between the general and the particular solution of the differential equation? Explain geometric visualization of the general and the particular solution for a differential equation!

3. What are the steps for the solving of the task from applied mathematics, natural sciences, and technology?