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<tr><td>Lesson title</td><td colspan="2">Mean Value Theorem</td></tr>
<tr><td>Learning objectives</td><td>- State and explain Rolle’s Theorem
- State and explain the Mean Value Theorem (MVT)
- Apply the MVT to velocity problems
- Use the MVT for explaining some of the basic facts of calculus</td><td rowspan="3">Strategies/Activities
☐Graphic Organizer
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<tr><td>Aim of the lecture / Description of the practical problem</td><td>The aim of the lecture is to present the Mean Value Theorem and to show why it is one of the most important results in calculus. Students will learn how it is connected to other known facts and will apply it to velocity problems.

Practical problem A car travelled 160 km in 2h. Show that the speedometer must have read 80 km/h at least once.</td></tr>
<tr><td>Previous knowledge assumed:</td><td>- Derivative of a function
- Increasing and decreasing functions
- Instantaneous and average velocity of a moving body
- Elementary functions and their derivatives</td></tr>
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Introduction / Theoretical basics	The most important theorem of calculus is the Mean Value Theorem, which is a simple consequence of **Rolle's Theorem**: Suppose that f is a continuous function on $[a,b]$ and differentiable on (a,b). If $f(a)=f(b)$, then there is some number c in (a,b) such that $f'(c)=0$. As a consequence, the zeroes of the derivative of a function separate the zeroes of the function. Rolle's Theorem can be applied to determine the number of zeroes of a function, by looking at the zeroes of its derivative.	sample ☐Conference ☐Check list ☐Diagnostics **Assessment as learning** ☐Self-assessment ☐Peer-assessment ☐Presentation ☐Graphic Organizer ☐Homework **Assessment of learning** ☐Test ☐Quiz ☐Presentation ☐Project ☐Published work
Action	**Example 1 for Rolle's Theorem** The equation $x^3+5x+1=0$ has one real solution since the derivative of $f(x)=x^3+5x+1$ is $f'(x)=3x^2+5$ and $3x^2+5=0$ doesn't have real solution. **Historical note. Michel Rolle (1652-1719)** was a French elementary school teacher, who was deeply interested in mathematics. He solved several problems of note, but did not prove the theorem that bears his name. In fact, in *Traité d'algebra,* 1690, he gave the special case of the theorem for the zeroes of a real polynomial. In that book Rolle introduced the notation $\sqrt[n]{x}$ for the *n*th root of a real number x. We will use Rolle's Theorem to prove the next central result of calculus: the Mean Value Theorem. This theorem states that when a function f is continuous on $[a,b]$ and differentiable on (a,b) there must be at least one point on the graph at which the slope of the tangent line is the same as the slope of the secant line through the points $(a, f(a))$ and $(b, f(b))$. The word *mean* refers to an average, that is, the value of the derivative at some point is the same as the average rate of change of the function on the interval. **Mean Value Theorem (MVT):** Suppose that f is a continuous function on $[a,b]$ and differentiable on (a,b). Then there is some number c in (a,b) such that $$f'(c)=\frac{f(b)-f(a)}{b-a}.$$ This theorem can be interpreted geometrically.	

Since the slope of the secant line through the points $A(a, f(a))$ and $B(b, f(b))$ is $\dfrac{f(b)-f(a)}{b-a}$, the MVT states that there must be at least one point on the graph of f at which the tangent line is parallel to the secant line through the points $A(a, f(a))$ and $B(b, f(b))$.

Proof. We define an auxiliary function

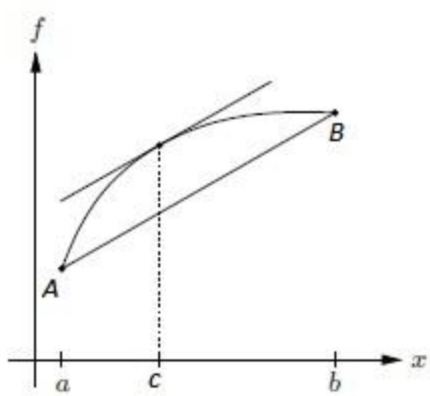
$$F(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a).$$

F is continuous function on $[a,b]$ and differentiable on (a,b). Since $F(a) = f(a) = F(b)$, Rolle's Theorem implies there is some c in (a,b) such that $F'(c) = 0$. Now

$$F'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

and so $F'(c) = 0$ is the same as

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$



Historical note. The Mean Value Theorem was first formulated

by **Joseph Louis Lagrange** (1736-1813), born Giuseppe Lodovico Lagrangia in Turin in a family of French heritage. As a young mathematician, starting from 1754, self-taught, he worked on the tautochrone problem and developed a new formal method
of maximizing and minimizing functionals in a way similar to finding extrema of functions. Lagrange wrote several letters to Leonhard Euler between 1754 and 1756 describing his results and Euler recognized his talent. In 1766, at the invitation of Frederick the Great, Lagrange succeeded Euler as Director of Mathematics at the Berlin Academy and in 1787, when Frederick died, Lagrange accepted King Louis XVI's invitation to Academie des Sciences in Paris.

The Mean Value Theorem is very useful in proving other theorems. We recall that the derivative of a constant function is zero.The converse of this result is proved via MVT.

Example 2. Determination of c in MVT for quadratic functions.

Let $f(x)=\alpha x^2+\beta x+\gamma$ with $\alpha \neq 0$ and let c be a number that satisfies the Mean Value Theorem over the interval $[a,b]$.
Find c.
Solution. Finding the c for quadratic functions is an easy calculation: it is unique and it is $c=\frac{a+b}{2}$. The result is surprising since c is always the midpoint of the interval $[a,b]$.

Example 3. Determination of c in MVT for square root function.

Let $f(x)=\sqrt{2x-4}$ and let c be a number that satisfies the Mean Value Theorem over $2 \leq x \leq 10$. Find c.
Solution. $f'(x)=1/\sqrt{2x-4}$ and by the MVT there is c in the interval such that

$$f'(c)=\frac{f(10)-f(2)}{10-2}=\frac{1}{2}.$$

From $f'(c)=1/\sqrt{2c-4}=1/2$ we find that $c=4$.

Question for the students for Examples 2 and 3:
- What is the derivative of quadratic and square root function?

- How do you solve quadratic equation?

The Mean Value Theorem is one of the most important theorems in calculus because it is very useful in proving other theorems. We recall that if $f(x)=k$ is a constant function, then $f'(x)=0$.
The converse of this result is proved in the next theorem.

Theorem If $f'(x)=0$ for all x in an interval $[a,b]$, then $f(x)$ is a constant on the interval.

Increasing and Decreasing Test

As an application of the Mean Value Theorem we can derive the following test for increasing and decreasing differentiable functions on an interval using their derivative.

Let f be a continuous function on $[a,b]$ and differentiable on (a,b).

(a) If $f'>0$ on (a,b), then f is increasing on $[a,b]$.

(b) If $f'<0$ on (a,b), then f is decreasing on $[a,b]$.

Proof. Let x_1 and x_2 be two numbers in $[a,b]$ such that $x_1<x_2$. By the Mean Value Theorem there is a number c in the interval (x_1,x_2) such that

$$f'(c)=\frac{f(x_2)-f(x_1)}{x_2-x_1}.$$

Now $f'(c)>0$ by assumption and $x_2-x_1>0$. Hence,

$$f(x_2)-f(x_1)>0 \text{ or } f(x_1)<f(x_2).$$

This shows that f is increasing.
Part (b) is proved similarly.

Example 4. Find where the function

$$f(x)=\frac{x}{1+x^2}$$

is increasing and where it is decreasing.

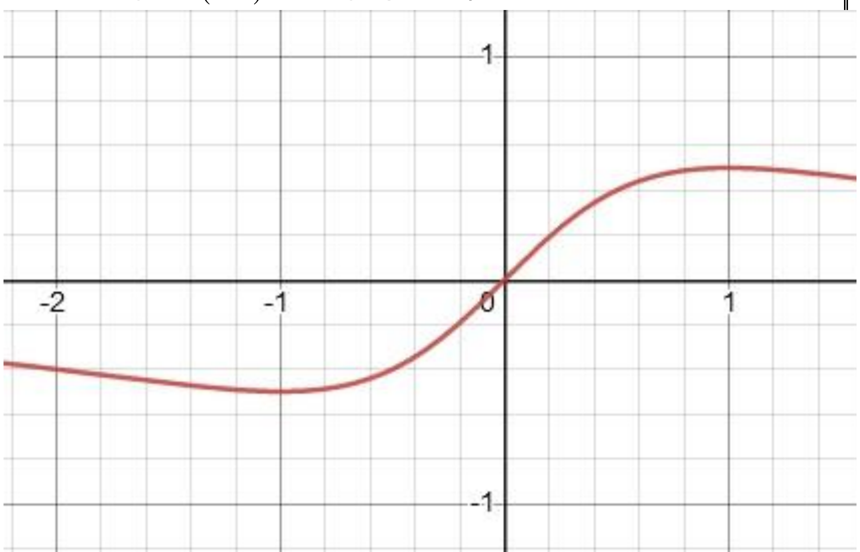
Solution. The derivative of the function is

$$f'(x)=\frac{1-x^2}{\left(1+x^2\right)^2}.$$

To use the Increasing and Decreasing Test, we have to know where $f'>0$ and where $f'<0$. We see that $f'>0$ for $-1<x<1$;

$f'<0$ for $x<-1$ and $x>1$.

So f is decreasing on $(-\infty,-1)$, increasing on $(-1,1)$ and decreasing on $(1,\infty)$. The graph of f below confirms this.



Applications.

Since the Mean Value Theorem has also kinematic interpretation, it can be applied to velocity problems.

Example 5. If an object moves in a straight line with position functions $s=f(t)$, then the average velocity between $t = a$ and $t = b$ is

$$\frac{f(b)-f(a)}{b-a}$$

and the velocity at $t = c$ is $f'(c)$. Thus, the Mean Value Theorem tells us that at some time $t = c$ between a and b, the

	instantaneous velocity $f'(c)$ is equal to that average velocity. **Solution of the Practical problem:** If a car travelled 160 km in 2h, the average velocity is 80 km/h. Applying the Mean Value Theorem we have that at some point during the travel, the speedometer which shows the instantaneous velocity must have read 80 km/h at least once. **Homework for students:** **1.** Find the result known as Extended Mean Value Theorem Apply it to derive the L'Hospital rule.	

Materials / equipment / digital tools / software	As a teaching material we use the textbooks from the references; any good book on calculus can also be used. For equipment in the classroom we need the usual black board and chalks. Digital tools that are used: graphic calculators, laptop, projector, smart board. Software: Mathematica, Geogebra can be used for drawing functions.	
Consolidation	• Use of materials, equipment, digital tools, software by teachers and students; • The teacher's discussion with the students through appropriate questions; • Independent solving of simple tasks by the students under the supervision of the teacher; • Given of examples by the teacher for introducing a new concept in a	

	cooperation and a discussion with the students; • Assignment of homework by the teacher with a time limit until the next class •

Reflections and next steps	
Activities that worked	**Parts to be revisited**
After the class, the teacher according to his personal perceptions regarding the success of the class fills in this part.	Through the success of the homework done by the students, questions and discussion at the beginning of the next class, the teacher comes to the conclusion which parts of this class should be revised.

References

J. Stewart: Calculus – Early Transcendentals, Thomson 2008
M. Lukarevski: Mathematics for computer scientists (in Macedonian), Univ. 'Goce Delcev' – Stip, 2019